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One Method for Proving Some Classes of Exponential Analytical Inequalities

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Abstract. In this paper we propose a method for proving some exponential inequalities based on power series expansion and analysis of derivations of the corresponding functions. Our approach provides a simple proof and generates a new class of appropriate inequalities, as well as allows direct establishment of the dependence between (the exponent of) some functions that occur as bounds of the approximation and the interval in which the corresponding inequality holds true.

1. Introduction and preliminaries

Papers [5]-[18] have recently considered various methods for proving mixed trigonometric polynomial inequalities of the form:

$$f(x) = \sum_{i=1}^{n} \alpha_i x^{p_i} \sin^{q_i} x \cos^{r_i} x > 0,$$

where $p_i, q_i, r_i \in \mathbb{N}_0$, $\alpha_i \in \mathbb{R} \setminus \{0\}$ and $x \in (0, \frac{\pi}{2})$. In monographs [3] and [4] were stated various analytical inequalities that can be reduced to mixed trigonometric polynomial inequalities. An algorithm that reduces proving of such inequalities to proving of the corresponding polynomial inequalities is developed in [9]. It is shown that many open problems and various inequalities recently published in renowned journals can be proved using the proposed algorithm.

In this paper we consider some exponential analytical inequalities whose proving can be reduced to analysis of power series expansion of the corresponding functions, and analysis of the derivatives of the function. We propose a method for generating and proving an appropriate class of exponential inequalities.

The starting point in our analysis are the following two (recently published) results:

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Statement 1.1. ([1] , *Theorem* 1) *If* $x \in (0,1)$ *then*

$$e^{-ax^2} < \cos x < e^{-x^2/2} \tag{1}$$

where $a \approx 0.615626$.

Statement 1.2. ([2] , *Theorem* 3) *If* $x \in (0,1)$ *then*

$$e^{-bx^2} < \frac{x}{\tan x} < e^{-x^2/3} \tag{2}$$

with the best possible constants $b \approx 0.443023$ *and* $1/3$.

In this paper we present and provide proof for an improvement of the previous statements.

Let us recall some of the well-known power series expansions that will be used in our proofs.

Based on [19], the following power series expansions hold:

$$\log\cos x = -\sum_{k=1}^{\infty} \frac{2^{2k-1}(2^{2k}-1)|\mathbf{B}_{2k}|}{k(2k)!}x^{2k}, \qquad (-\pi/2 < x < \pi/2), \tag{3}$$

and

$$\log\frac{x}{\tan x} = -\sum_{k=1}^{\infty} \frac{(2^{2k-1}-1)|\mathbf{B}_{2k}|}{k(2k)!}x^{2k}, \qquad (0 < x < \pi/2), \tag{4}$$

where $\mathbf{B}_i$ $(i \in \mathbb{N})$ are Bernoulli's numbers.

2. Main results

First, we consider the relationship between the number of zeros and the number of (local) minimums of a real function, as well as some properties of its derivatives.

We prove the following assertion:

Theorem 2.1. *Let* $f : (0,c) \longrightarrow \mathbb{R}$ *be* m *times differentiable function (for some* $m \geq 2$, $m \in \mathbb{N}$ *) which satisfies the following conditions:*

(a) $f^{(m)}(x) > 0$ *for* $x \in (0,c)$;

(b) *there is a right neighbourhood of zero in which the following inequalities hold true:*

$$f(x) < 0,\ f'(x) < 0,\ \ldots,\ f^{(m-1)}(x) < 0;$$

(c) *there is a left neighbourhood of* c *in which the following inequalities hold true:*

$$f(x) > 0,\ f'(x) > 0,\ \ldots,\ f^{(m-1)}(x) > 0.$$

Then the function f *has exactly one zero* $x_0 \in (0,c)$, *and* $f(x) < 0$ *for* $x \in (0,x_0)$ *and* $f(x) > 0$ *for* $x \in (x_0,c)$. *Also, the function* f *has exactly one local minimum in the interval (0, c) i.e. there is exactly one point* $t_0 \in (0,x_0) \subset (0,c)$ *such that* $f(t_0) < 0$ *is the minimal value of function* $f(x)$ *over the interval* $(0,x_0)$ *i.e.* $(0,c)$.

Proof. On the basis of condition (a), as $f^{(m)}(x) > 0$ for $x \in (0,c)$, it follows that $f^{(m-1)}(x)$ is a monotonically increasing function for $x \in (0,c)$. Based on conditions (b) and (c), we conclude that there exists exactly one zero $x_{m-1} \in (0,c)$ of the function $f^{(m-1)}(x)$. Next, we can conclude that function $f^{(m-2)}(x)$ is monotonically decreasing for $x \in (0,x_{m-1})$ and monotonically increasing for $x \in (x_{m-1},c)$. It is clear that the function $f^{(m-2)}(x)$ has exactly one minimum in the interval $(0,c)$ at point x_{m-1}, and so $f^{(m-2)}(x_{m-1}) < 0$ holds. On the basis of condition (c), it follows that function $f^{(m-2)}(x)$ has exactly one root x_{m-2} in the interval $(0,c)$ $(x_{m-2} \in (x_{m-1},c))$ and $f^{(m-2)}(x) < 0$ for $x \in (0,x_{m-2})$ as well as $f^{(m-2)}(x) > 0$ for $x \in (x_{m-2},c)$.
By repeating the above procedure, we get the assertion given in the theorem. □

Remark 2.2. *The Theorem 2.1 is a natural extension of Theorem* 3, *from* [18], *which was applied in* [18] *to inequalities involving the sinc function.*

2.1. Generalization of Statement 1.1

Consider inequalities (1) for $x \in (0, \pi/2)$ and parameter $a \in \mathbb{R}^+$.

It is enough to analyse the following equivalent inequalities:

$$-ax^2 < \log\cos x < -x^2/2, \tag{5}$$

for $x \in (0, \pi/2)$ and parameter $a \in \mathbb{R}^+$.

Firstly, we prove that the right-hand side of inequality (5) holds true for every $x \in (0, \pi/2)$.

The real analytical function

$$f_1(x) = -\tfrac{1}{2}x^2 - \log\cos x : \left(0, \tfrac{\pi}{2}\right) \longrightarrow \mathbb{R},$$

has the following power series expansion

$$f_1(x) = \sum_{k=2}^{\infty} \frac{2^{2k-1}(2^{2k}-1)|\mathbf{B}_{2k}|}{k(2k)!} x^{2k},$$

for $x \in (0, \pi/2)$, from which it follows that $f_1(x) > 0$ for $x \in (0, \pi/2)$.

Consider now the left-hand side of inequality (5). The corresponding real analytical function

$$f_2(x) = -ax^2 - \log\cos x : \left(0, \tfrac{\pi}{2}\right) \longrightarrow \mathbb{R}$$

has the following power series expansion:

$$f_2(x) = \left(\frac{1}{2} - a\right)x^2 + \sum_{k=2}^{\infty} \frac{2^{2k-1}(2^{2k}-1)|\mathbf{B}_{2k}|}{k(2k)!} x^{2k}. \tag{6}$$

Based on the above power series expansion, it is not hard to show that for $a > 1/2$ and $m = 3$ the conditions of Theorem 2.1 are fulfilled. Therefore, in the interval $\left(0, \frac{\pi}{2}\right)$ the function $f_2(x)$ has exactly one zero x_0, and there is exactly one point $t_0 \in (0, x_0)$ such that $f_2(t_0) < 0$ is minimal value of the function $f_2(x)$ over the interval $(0, x_0) \subset (0, \pi/2)$.

For every fixed $x_0 \in \left(0, \frac{\pi}{2}\right)$ and

$$a = -\frac{\log\cos x_0}{x_0^2},$$

we have $f_2(x_0) = 0$, i.e. x_0 is the (unique) zero of the function $f_2(x)$, and for all $x \in (0, x_0)$ $f_2(x) < 0$. Further, based on (6) and $f_2(x_0) = 0$ we have:

$$a = \frac{1}{2} + \sum\nolimits_{k=2}^{\infty} \tfrac{2^{2k-1}(2^{2k}-1)|\mathbf{B}_{2k}|}{k(2k)!} x_0^{2k-2} > \frac{1}{2}.$$

Overall, on the basis of the previous consideration, the following improvement of Statement 1.1 has been proved:

Theorem 2.3. *Let $x_0 \in (0, \pi/2)$. Then for every $x \in (0, x_0)$ the following inequalities hold true*

$$e^{-ax^2} < \cos x < e^{-x^2/2}$$

with the best possible constants $a = -\left(\log\cos x_0\right)/x_0^2$ and $1/2$.

Remark 2.4. *For $x_0 = 1$ we get Statement* 1.1. *and constant $a = -\log\cos 1 \approx 0.615626$. In that case, $t_0 \approx 0.736713$ and the minimum value $f_2(t_0) \approx -0.0339435$ determines the absolute value of the largest error $\delta \approx 0.0339435$ in the approximation of the function* $\log\cos x$ *with the square polynomial $-ax^2$ over interval* $(0, 1)$.

2.2. Generalization of Statement 1.2

Consider inequalities (2) for $x \in (0, \pi/2)$ and parameter $b \in \mathbb{R}^+$.

It is enough to analyse the following equivalent inequalities:

$$-bx^2 < \log \frac{x}{\tan x} < -\frac{x^2}{3} \tag{7}$$

for $x \in (0, \pi/2)$ and parameter $b \in \mathbb{R}^+$.

For the right-hand side of inequality (7) it is enough to observe the real analytical function

$$g_1(x) = -\tfrac{1}{3}x^2 - \log \frac{x}{\tan x} : \left(0, \tfrac{\pi}{2}\right) \longrightarrow \mathbb{R},$$

and the corresponding power series expansion:

$$g_1(x) = \sum_{k=2}^{\infty} \frac{2^{2k}(2^{2k-1}-1)|\mathbf{B}_{2k}|}{k(2k)!} x^{2k},$$

for $x \in (0, \pi/2)$, from which it follows that $g_1(x) > 0$ for $x \in (0, \pi/2)$.

Consider now the left-hand side of inequality (7). The corresponding real analytical function

$$g_2(x) = -bx^2 - \log \frac{x}{\tan x} : \left(0, \tfrac{\pi}{2}\right) \longrightarrow \mathbb{R}$$

has the following power series expansion:

$$g_2(x) = \left(\frac{1}{3} - b\right)x^2 + \sum_{k=2}^{\infty} \frac{2^{2k}(2^{2k-1}-1)|\mathbf{B}_{2k}|}{k(2k)!} x^{2k}. \tag{8}$$

It is not hard to show that for $b > 1/3$ and $m = 3$ the conditions of Theorem 2.1 are fulfilled. Therefore, the function $g_2(x)$ has exactly one zero x_0 in the interval $\left(0, \frac{\pi}{2}\right)$, and there is exactly one point $t_0 \in (0, x_0)$ such that $g_2(t_0) < 0$ is the minimal value of the function $g_2(x)$ over the interval $(0, x_0) \subset (0, \pi/2)$.

For every fixed $x_0 \in \left(0, \frac{\pi}{2}\right)$ and

$$b = -\frac{\log \frac{x_0}{\tan x_0}}{x_0^2},$$

we have $g_2(x_0) = 0$, i.e. x_0 is the (unique) zero of the function $g_2(x)$, and for all $x \in (0, x_0)$ $g_2(x) < 0$. Further, based on (8) and $g_2(x_0) = 0$ we have:

$$b = \frac{1}{3} + \sum\nolimits_{k=2}^{\infty} \tfrac{2^{2k}(2^{2k-1}-1)|\mathbf{B}_{2k}|}{k(2k)!} x_0^{2k-2} > \frac{1}{3}.$$

Overall, on the basis of the previous consideration, the following improvement of Statement 1.2 has been proved:

Theorem 2.5. *Let $x_0 \in (0, \pi/2)$. Then for every $x \in (0, x_0)$ the following inequalities hold true*

$$e^{-bx^2} < \log \frac{x}{\tan x} < e^{-x^2/3}$$

with the best possible constants $b = -\left(\log \frac{x_0}{\tan x_0}\right)/x_0^2$ and $1/3$.

Remark 2.6. *For $x_0 = 1$ we get Statement 1.2 and constant $b = \log\tan 1 \approx 0.443023$. In that case, $t_0 \approx 0.737815$ and the minimum value $g_2(t_0) \approx -0.0324168$ determines the absolute value of the largest error $\delta \approx 0.0324168$ in the approximation of the function $\log \frac{x}{\tan x}$ with the square polynomial $-bx^2$ over interval $(0, 1)$.*

3. Conclusion

In this paper we established some exponential analytic inequalities, using the results of Theorem 2.1, as well as the power series expansion of the corresponding functions. In particular, we proved that the following inequalities hold true:

$$e^{-ax^2} < \cos x < e^{-x^2/2},$$

$$e^{-bx^2} < \log \frac{x}{\tan x} < e^{-x^2/3}$$

for every $x \in (0, \alpha)$, where $0 < \alpha < \pi/2$, with the best possible constants $a = -(\log \cos \alpha) / \alpha^2$ and $1/2$, and $b = -\left(\log \frac{\alpha}{\tan \alpha}\right) / \alpha^2$ and $1/3$, respectively.

Our approach provides a simple proof and allows direct establishment of the dependence between the exponents of the bounds of approximation and the interval in which the corresponding inequality holds true. In this way, we get a class of inequalities as well as intervals in which these inequalities are true. We believe that our method can contribute to broader application of the power series in the study of inequalities.

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