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Filtering Method for Linear and Non-Linear Stochastic Optimal Control Of Partially Observable Systems II

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Abstract. In this paper we studied stochastic optimal control problem based on partially observable systems (SOCPP) with a control factor on the diffusion term. A SOCPP has state and observation processes. This kind of problem has also a minimum payoff function. The payoff function should be minimized according to the partially observable systems consist of the state and observation processes. In this regard, the filtering method is used to evaluat this kind of problem and express full consideration of it. Finally, presented estimation methods are used to simulate the solution of a partially observable system corresponding to the control factor of this problem. These methods are numerically used to solve linear and nonlinear cases.

Keywords: Stochastic optimal control; Control on diffusion term; Partially observation system; Filtering method.

1. Introduction

Optimal control was considered in the late 1950s. This science is used in many fields. For example, physics, engineering, finance, chemistry and others [8, 14]. It plays an efficient role in these sciences and has had a lot of developments over the time.

One of the important and efficient branch of this science is stochastic optimal control. This branch is interesting, because everything has a random scheme. Stochastic optimal control (SOC) is studied in various cases [4]. The HJB method is more popular than the others in solving this kind of problem [8]. Many studies have been done about this subject. For example, in [11] Kafash et al used this equation to evaluate this kind of problem. One of the important and useful case of SOC problem is SOC with jumping diffusion. It has been interested in identifying capital markets [17]. For example, in the classical Markovian case with SOC problem of jumping diffusions, the equation is reduced to the classical HJB equation [18]. In [19], numerical method is presented to estimate the HJB equation.

An optimal control problem (OCP) is solved by various methods. One of the most popular form of these methods is the use of HJB equation [9]. Many authors have studied this equation. For example, in [12] this equation is estimated by Chebyshev polynomials and in [13], Variational iteration method is used to estimate the HJB equation. There are also many other investigations in this case.

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One of the most important and popular branch of stochastic optimal control is the SOCPP. Many studies have been done in this field. For example, in [26] the maximum principle for a SOCPP is used to obtain the control factor. In [23], by finding the relation between the adjoint process and maximum principle, the control factor is obtained. Some studies are interested in the field of finance. For example, in [20] by partially observing of stock market and using SOCPP principles, a model for portfolio optimization is found.

In this paper we study linear and nonlinear cases of a SOCPP with a control factor on the diffusion part. In fact, it is the second edition of [7] which focuses on SOCPP with a control factor on the diffusion part. We extend the studies of [3, 25] to the control diffusion SOCPP and numerical estimation methods. At first, we introduce the general form of this kind of problem. A SOCP problem has a minimum payoff function and two SDEs named the state and observation processes. The control factor of this kind of problem should be found. So, present the full description of finding optimal control. In this way, select a general form of control factor as a feedback control. The existence and uniqueness of these feedback controls are evaluated in this paper. Numerical methods are presented in a linear filtering mode to evaluate the corresponding partially observable system. Finally, introduce numerical methods to estimat this kind of problem.

Optimal control problems based on partially observable information are always hard to study. One difficulty of this problem is that the classical separation principle does not hold true generally. In 2008, a backward separation method was proposed by Wang and used to overcome the difficulty [24]. For more details, Wang, Wu and Xiong have studied this issue. They worked on forward and backward stochastic differential equations and used a filtering method to find the optimal control [25]. In this study, the forward and backward stochastic differential equations are very similar to the system in our work and the forms of optimal control are the same. It is remarkable that, the backward separation method used in [25] is available for the control problem studied in this paper.

The novel contribution of this paper modifies the Bensoussan's study [3, 25] on the SOCPP with a control factor on the diffusion term and solves nonlinear problems by presenting a method. Furthermore, simulations show the efficiency of this presented method.

2. Preliminaries

In this section all the necessary concepts about a SOCPP with a control factor on the diffusion term are introduced [1–3]. At first, consider (Ω, A, P) as a probability space in which all the events occur. Define the state and observation processes as follows,

$$\begin{cases} dx = \Big(F(t)x + G(t)v(t) + f(t)\Big)dt + \sum_{i=1}^{d}\Big(B_i(t)x + D_i(t)v(t) + g_i(t)\Big)dw_i(t) + \sigma_0 dw(t), \\ x(0) = \zeta, \\ dy = \Big[H(t)x + h(t)\Big]dt + \sigma_1 db(t), \\ y(0) = 0. \end{cases} \tag{1}$$

In 1, x is the 'state process', the 'observation process' is denoted by y and the initial value ζ is a vector of random variable with Gaussian probability law ,mean x_0 and noise covariance matrix P_0. The processes $w(\cdot)$ and $b(\cdot)$ are wiener processes with the noise covariance matrices $Q(\cdot)$ and $R(\cdot)$ respectively. The wiener processes $w_1, \dots, w_d$ are scalar and standard. The noise covariance $R(\cdot)$ is a uniformly positive definite matrix. It should be mentioned that, the variable ζ and the processes $w_i(\cdot)$, $w(\cdot)$ and $b(\cdot)$ are mutually independent for all $i = 1, \dots, d$. All vector functions $F, G, f, B_i, D_i, g_i, H$ and h are bounded [3]. The vector functions $F, x, G, v, f, B_i, D_i, g_i, y, H, h, w_i(\cdot), w(\cdot), b(\cdot), \sigma_0$ and σ_1 are $n \times n, n \times 1, n \times n, n \times 1, n \times 1, n \times n, n \times n, n \times 1, k \times 1, k \times n, k \times 1, 1 \times 1, m \times 1, k \times 1, n \times m$ and $k \times k$ dimensional respectively.

The corresponding noise covariance matrices $Q(\cdot)$ and $R(\cdot)$ are defined by $Q = (\sigma_0 \times \sigma_0^*)_{n \times n}$ and $R = (\sigma_1 \times \sigma_1^*)_{k \times k}$ respectively [8, 16].

Now introduce the partially observable stochastic optimal control problem with a control factor on the diffusion term of the state process as follows:

Suppose that, $\nu(\cdot) \in L^2(0,T;R^n)$, $x(\cdot) \in L^2(0,T;R^n)$ and $z(\cdot) \in L^2(0,T;R^k)$. This kind of problem has the state and observation processes 1 and a payoff function in the following form:

$$J(\nu(\cdot)) = E\Big\{\int_0^T (x^*Mx + \nu^*N\nu + 2mx + 2n\nu)dt + x^*(T)M_Tx(T) + 2m_Tx(T)\Big\}. \tag{2}$$

where $M(\cdot)$ is symmetric non negative, $N(\cdot)$ is symmetric uniformly positive definite and all the time functions M, N, m, and n are bounded.

2.1. Framework

To have necessary definitions for admissibility of control factor, define the 'without control' and the 'difference processes' from 1. These processes are the separation of 1 that help us to find control factors. Consider the state and observation processes from the relation 1 and define the new processes which called 'without control processes' as follows:

$$\begin{cases} d\alpha = \Big(F(t)\alpha + f(t)\Big)dt + \sum_{i=1}^d \Big(B_i(t)\alpha + g_i(t)\Big)dw_i(t) + \sigma_0(t)dw(t), \\ \alpha(0) = \zeta, \\ d\beta = \Big[H(t)\alpha + h(t)\Big]dt + \sigma_1(t)db(t), \\ \beta(0) = 0. \end{cases} \tag{3}$$

The processes $\alpha(\cdot)$ and $\beta(\cdot)$ are according to the state and observation processes respectively [3, 6, 8].

Assume that $\nu(t) \in L^2(0, T \times \Omega, dt \otimes dP; R^k)$ is a square integrable process. Then define the 'difference processes' x_1 and y_1 as follows:

$$\begin{cases} dx_1 = d(x-\alpha) = (Fx_1 + G\nu)dt + \sum_{i=1}^d \Big(B_i(t)x_1 + D_i(t)\nu(t)\Big)dw_i(t), \\ x_1(0) = 0, \\ dy_1 = d(y-\beta) = Hx_1dt, \\ y_1(0) = 0. \end{cases} \tag{4}$$

The above relations are solvable for each sample path [25].

The state and observation processes x and y in separative mode which defined for any control ν are satisfied in the following new relations:

$$x(t) = \alpha(t) + x_1(t). \tag{5}$$

$$y(t) = \beta(t) + y_1(t). \tag{6}$$

First, consider the family of σ - algebras Z_ν^t, consists of all the information (until time t) from the observation process $y(\cdot)$.

$$Z_\nu^t = \sigma(y(s), s \leq t). \tag{7}$$

The optimal control is selected in the space of admissible controls. The definitions of admissibility are as follows:

Definition 2.1. *(natural definition) Control $\nu(\cdot)$ is admissible if and only if $\nu(t)$ is adapted to Z_ν^t.*

After that, the family of σ - algebras F^t, consists of all the information (until time t) from without control observation process $\beta(\cdot)$ is introduced.

$$F^t = \sigma(\beta(s), s \leq t). \tag{8}$$

Corollary 2.2. *If control $\nu(\cdot)$ is admissible then $F^t \subset Z^t_\nu$.*

Proof. (Please see the definition of adaptability and measurement in [8].)

First we define the σ - algebras $Y^t_\nu = \sigma(y_1(s), s \leq t)$. We can conclude the relation $F^t \subset (Z^t_\nu \vee Y^t_\nu)$ from the relation 6 . If the definition 2.1 is established and according to the definition of adaptability, the control $\nu(\cdot)$ is Z^t_ν - measurable and the relation $Y^t_\nu \subset Z^t_\nu$ is satisfied. Therefore, $F^t \subset Z^t_\nu$. □

Definition 2.3. *(more restrictive definition) Control $\nu(\cdot)$ is admissible if and only if $\nu(t)$ is adapted to F^t and to Z^t_ν.*

Corollary 2.4. *If $\nu(\cdot)$ is admissible, then $F^t = Z^t_\nu$.*

It is one of the important conclusion of expression 6. This conclusion makes it easy to obtain the Kalman filter $\hat{x}$ as follows:

$$\hat{x}(t) = E[x(t)|Z^t_\nu] = E[x(t)|F^t]. \tag{9}$$

Proof. (Please see the definition of adaptability and measurement in [8].)

According to the definition of adaptability, If the definition 2.3 is established then $\nu(t)$ is Z^t_ν - measurable. So, we can conclude the following statment:
$\forall \ C \in Z^t_\nu, \quad \exists \ A \in \mathfrak{B}(\mathbb{R}^m) \ \ s.t \ \ \nu^{-1}(A) = C.$
Now based on this definition, the control $\nu(t)$ is also F^t - measurable. Hence, $C = \nu^{-1}(A)$ is also in F^t. Therefore, $Z^t_\nu \subset F^t$. According to the corollary 2.2 and $Z^t_\nu \subset F^t$, it can be concluded that $F^t = Z^t_\nu$. So, the relation 9 is established. □

Define the 'innovation process' $I(\cdot)$ as follows to make a control on the filtering processes [3].

$$I(t) = \beta(t) - \int_0^T (H\hat{\alpha} + h)ds = y(t) - \int_0^T (H\hat{x} + h)ds. \tag{10}$$

The equality 10 is according to the second relation of 3 and 1 as follows:

$$\beta(t) - \underbrace{\beta(0)}_{0} = \int_0^t [H(t)\alpha + h(t)]dt + \int_0^t \sigma_1 db$$
$$\xrightarrow{9} I(t) = \int_0^t \sigma_1 db = \beta(t) - \int_0^t [H(t)\hat{\alpha} + h(t)]dt.$$

$$Z(t) - \underbrace{Z(0)}_{0} = \int_0^t [H(t)x + h(t)]dt + \int_0^t \sigma_1 db$$
$$\xrightarrow{9} I(t) = \int_0^t \sigma_1 db = Z(t) - \int_0^t [H(t)\hat{x} + h(t)]dt.$$

The 'innovation process' $I(t)$ is a F^t wiener process with covariance matrix $R(\cdot)$. The control factor is defined by two parameters Λ and λ in the linear feedback form as follows:

$$\nu(t) = \Lambda(t)x + \lambda(t). \tag{11}$$

Now impose 11 on the state process of system 1.

$$dx = \Big(\Big[F(t) + G(t)\Lambda(t)\Big]x + G(t)\lambda(t) + f(t)\Big)dt + \textstyle\sum_{i=1}^{d}\Big(\Big[B_i(t) + D_i(t)\Lambda(t)\Big]x + D_i(t)\lambda(t) + g_i(t)\Big)dw_i(t) + \sigma_0 dw(t). \tag{12}$$

The following theorem is about the admissibility of control in the sense of definition 2.3. It is one of the innovative cases of this article. The differential equation of this theorem can be obtained by using the methods of [9].

Theorem 2.5. *(Admissible control) If $v(\cdot)$ is admissible in the sense of definition 2.3, then the Kalman filter $\hat{x}(t)$ is the solution of the following differential equation,*

$$\begin{cases} d\hat{x} = \Big(F(t)\hat{x} + G(t)v(t) + f(t)\Big)dt + \sum_{i=1}^{d}\Big(B_i(t)\hat{x} + D_i(t)v(t) + g_i(t)\Big)dw_i(t) + PH^*R^{-1}\Big(dz - (H\hat{x} + h)dt\Big), \\ \hat{x}(0) = x_0, \end{cases} \tag{13}$$

where P is the solution of bellow ordinary differential equation.

$$\begin{cases} \frac{dP}{dt} + F^*P + PF + \sum_{i=1}^{d} B_i^*PB_i + Q - \Big(PG + \sum_{i=1}^{d} B_i^*PD_i\Big)\Big(R + \sum_{i=1}^{d} D_i^*PD_i\Big)^{-1}\Big(G^*P + \sum_{i=1}^{d} D_i^*PB_i\Big) = 0, \\ P(0) = E\Big[\Big(x_0 - E(x_0)\Big)\Big((x_0 - E(x_0)\Big)^*\Big]. \end{cases} \tag{14}$$

Proof. From 5 we have the following relation:

$$\hat{x}(t) = x_1(t) + E[\alpha(t)|F^t] = x_1(t) + \hat{\alpha}(t). \tag{15}$$

The process $\hat{\alpha}$ is the Kalman filter of process α in the system 3. This Kalman filter is defined as follows [21],

$$\begin{cases} d\hat{\alpha} = \Big(F(t)\hat{\alpha} + G(t)v(t) + f(t)\Big)dt + \sum_{i=1}^{d}\Big(B_i(t)\hat{\alpha} + D_i(t)v(t) + g_i(t)\Big)dw_i(t) + PH^*R^{-1}\Big(dz - (H\hat{\alpha} + h)dt\Big), \\ \hat{\alpha}(0) = x_0, \end{cases} \tag{16}$$

where P is the solution of 12. By using the relation 15 deduce the following differential equation:

$$d\hat{x} \;\; = \Big(F(t)\hat{x} + G(t)v(t) + f(t)\Big)dt + \textstyle\sum_{i=1}^{d}\Big(B_i(t)\hat{x} + D_i(t)v(t) + g_i(t)\Big)dw_i(t) + PH^*R^{-1}\Big(dz - (H\hat{\alpha} + h)dt\Big). \tag{17}$$

But from relations 5 and 6, find the following equalities:

$$d\beta - (H\hat{\alpha} + h)dt = dz - Hx_1dt - (H\hat{\alpha} + h)dt = dz - (H\hat{x} + h)dt. \tag{18}$$

□

Hence, the relation 13 is obtained by substituting 18 in 17.

The above theorem is according to the ideas of corresponding theorem in [3].
After finding the admissible controls we should choose the optimal one among them. The following theorem is about the optimality of the control on a SOCPP.

Theorem 2.6. *(Optimality of control on a SOCPP with a control factor on the diffusion term) For a SOCPP with a payoff 2 and processes 1, there is an optimal and unique feedback control in the form of 11 with the following control parameters:*

$$\Lambda = -\Big(N + \sum_{i=1}^{d} D_i^*\Pi D_i\Big)^{-1}\Big(G^*\Pi + \sum_{i=1}^{d} D_i^*\Pi B_i\Big), \tag{19}$$

$$\lambda = -\Big(N + \sum_{i=1}^{d} D_i^* \Pi D_i\Big)^{-1} \Big(n^* + G^* r + \sum_{i=1}^{d} D_i^* \Pi g_i\Big), \tag{20}$$

This feedback control is among the set of admissible controls in the sense of definition 2.3. Variables Π and r are the solutions of the following differential equations respectively.

$$\begin{cases} \frac{d\Pi}{dt} + F^*\Pi + \Pi F + \sum_{i=1}^{d} B_i^*\Pi B_i + M - \Big(\Pi G + \sum_{i=1}^{d} B_i^*\Pi D_i\Big)\Big(N + \sum_{i=1}^{d} D_i^*\Pi D_i\Big)^{-1}\Big(G^*\Pi + \sum_{i=1}^{d} D_i^*\Pi B_i\Big) = 0. \\ \Pi(T) = M_T. \end{cases} \tag{21}$$

$$\begin{cases} \frac{dr}{dt} + \Big[F^* - \Big(\Pi G + \sum_{i=1}^{d} B_i^*\Pi D_i\Big)\Big(N + \sum_{i=1}^{d} D_i^*\Pi D_i\Big)^{-1} G^*\Big] r + m^* + \Pi f + \sum_{i-1}^{d} B_i^*\Pi g_i \\ \quad - \Big(\Pi G + \sum_{i=1}^{d} B_i^*\Pi D_i\Big)\Big(N + \sum_{i=1}^{d} D_i^*\Pi D_i\Big)^{-1}\Big(n^* + \sum_{i=1}^{d} D_i^*\Pi g_i\Big) = 0. \\ r(T) = m_T^*. \end{cases} \tag{22}$$

Proof. Let 11 be the control factor. Now assume that the control parameters Λ and λ are unspecified in this form. Denote the control factor 11 by $u(\cdot)$ instead of $\nu(\cdot)$. The corresponding state and observation processes are denoted by $k(\cdot)$ and $\xi(\cdot)$ respectively.

$$\begin{cases} dk = \Big(F(t)k + G(t)\nu(t) + f(t)\Big)dt + \sum_{i=1}^{d} \Big(B_i(t)k + D_i(t)\nu(t) + g_i(t)\Big)dw_i(t) + \sigma_0 dw(t), \\ k(0) = \zeta, \\ d\xi = \Big[H(t)\xi + h(t)\Big]dt + \sigma_1 db(t), \\ \xi(0) = 0. \end{cases} \tag{23}$$

The Kalman filter of the state process corresponding to the observation process $\xi(\cdot)$ and control factor $u(t) = \Lambda(t)\hat{k} + \lambda(t)$ is as follows [15].

$$\begin{cases} d\hat{k} = \Big(\Big[F(t) + G(t)\Lambda(t)\Big]\hat{k} + G(t)\lambda(t) + f(t)\Big)dt + \sum_{i=1}^{d} \Big(\Big[B_i(t) + D_i(t)\Lambda(t)\Big]\hat{k} + D_i(t)\lambda(t) + g_i(t)\Big)dw_i(t) \\ \quad + PH^*R^{-1}\Big(d\xi - (H\hat{k} + h)dt\Big), \\ \hat{k}(0) = x_0. \end{cases} \tag{24}$$

To prove the optimality of this case, make 'change of control function' $\nu(\cdot)$. Suppose that process $\mu(\cdot)$ is adapted to F^t. Now if the process $\eta(\cdot)$ is adapted to F^t and has the bellow differential equation then make sure that the 'change of control function' $\nu(\cdot)$ is also adapted to F^t.

$$\begin{cases} d\eta = \Big(\Big[F(t) + G(t)\Lambda(t)\Big]\eta + G(t)\lambda(t) + f(t)\Big)dt + \sum_{i=1}^{d} \Big(\Big[B_i(t) + D_i(t)\Lambda(t)\Big]\eta + D_i(t)\lambda(t) + g_i(t)\Big)dw_i(t) \\ \quad + PH^*R^{-1}\Big(d\xi - (H\eta + h)dt\Big), \\ \eta(0) = x_0. \end{cases} \tag{25}$$

where the 'innovation process' is as follows,

$$dI = dy - (H\eta + h)dt, \tag{26}$$

and the change of control function is,

$$\nu(t) = \Lambda(t)\eta(t) + \lambda(t) + \mu(t). \tag{27}$$

In fact the state and observation processes are corresponding to the control factor $\nu(t)$ and $\eta(t)$ is the Kalman filter of process x as expressed in the expression 9 [2, 25]. Furthermore $\eta(t)$ is adapted to F^t. Impose the additional condition that process $\mu(t)$ is adapted to Z_ν^t, then it can be understood from the 'innovation process' 20 and equation 19 that $\eta(t)$ is also adapted to Z_ν^t. So the process $\eta(t)$ is adapted to the both families of σ - algebras F^t and Z_ν^t [3]. Therefore, $\nu(\cdot)$ is admissible in the sense of definition 2.3.

Consider the state (x) and observation (y) processes which are in the system 1. The Kalman filter of the state process x is defined according to the control $\nu(\cdot)$ as follows [1, 16]:

$$\begin{cases} d\hat{x} = \Big(\big[F(t) + G(t)\Lambda(t)\big]\hat{x} + G(t)\lambda(t) + G(t)\mu + f(t)\Big)dt \\ \quad + \sum_{i=1}^{d}\Big(\big[B_i(t) + D_i(t)\Lambda(t)\big]\hat{x} + D_i(t)\lambda(t) + D_i(t)\mu + g_i(t)\Big)dw_i(t) \\ \quad + PH^*R^{-1}\Big(d\xi - (H\hat{x} + h)dt\Big), \\ \hat{x}(0) = x_0. \end{cases} \tag{28}$$

By comparing two Kalman filters 28 and 24, find the following useful comments,

$$dy - (H\hat{k} + h)dt = d\xi - (H\hat{y} + h)dt = d\beta - (H\hat{\alpha} + h)dt\cdot \tag{29}$$

Now define the process $\tilde{x}(t)$ by subtracting these two Kalman filters.

$$d\tilde{x} = \Big(\big[F(t) + G(t)\Lambda(t) + G(t)\mu\big]\tilde{x}\Big)dt + \sum_{i=1}^{d}\Big(\big[B_i(t) + D_i(t)\Lambda(t) + D_i(t)\mu\big]\tilde{x}\Big)dw_i(t), \quad \tilde{x}(0) = 0. \tag{30}$$

Where,

$$\tilde{x}(t) = \hat{x}(t) - \hat{k}(t). \tag{31}$$

Now, by using Kalman filter rule 14 for the separation processes 5 and 6, find the following two new relations for $\hat{x}$ and $\hat{k}$ [10, 16].

$$\begin{cases} \hat{x}(t) = \hat{\alpha}(t) + x_1(t), \\ \hat{k}(t) = \hat{\alpha}(t) + k_1(t). \end{cases} \tag{32}$$

Now, due to the expressions 5 and 6 and the above relations find the following equations,

$$\begin{cases} d(x_1 - k_1) = \Big(\big[F(t) + G(t)\Lambda(t) + G(t)\mu\big]\big(x_1 - k_1\big)\Big)dt + \sum_{i=1}^{d}\Big(\big[B_i(t) + D_i(t)\Lambda(t) + D_i(t)\mu\big]\big(x_1 - k_1\big)\Big)dw_i(t), \\ x_1(0) - k_1(0) = 0. \end{cases} \tag{33}$$

From the above equation conclude that,

$$\hat{x}(t) - \hat{k}(t) = x_1(t) - k_1(t) = x(t) - k(t) = \tilde{x}(t). \tag{34}$$

To compute the value of cost functions, substitute $\hat{x}(t) = \tilde{x}(t) + \hat{y}(t)$ as a result of expression 25 in the cost functional $J(\nu(\cdot))$ and simplify it as follows:

$$\begin{aligned} J(\nu(\cdot)) = K(\mu(\cdot)) = E\Big\{\int_0^T \Big[\big(y(t) + \tilde{x}(t)\big)^* M\big(y(t) + \tilde{x}(t)\big) + \big(\Lambda\hat{y} + \lambda + \Lambda\hat{x} + \mu\big)^* N\big(\Lambda\hat{y} + \lambda + \Lambda\hat{x} + \mu\big) \\ + 2m\big(y + \tilde{x}\big) + 2n\big(\Lambda\hat{y} + \lambda + \Lambda\hat{x} + \mu\big)\Big]dt + \big(y(T) + \tilde{x}(T)\big)^* M_T\big(y(T) + \tilde{x}(T)\big) + 2m_T\big(y(T) + \tilde{x}(T)\big)\Big\} \end{aligned}$$

By simplifying the above relation the corresponding payoff function is:

$$J(\nu(\cdot)) = J(u(\cdot)) + E\Big\{\int_0^T \big[\tilde{x}^* M\tilde{x} + (\Lambda\tilde{x} + \mu)^* N(\Lambda\tilde{x} + \mu)\big]dt + \tilde{x}(T)^* M_T\tilde{x}(T)\Big\} + 2X,$$

where,

$$X = E\Big\{ \int \Big[\tilde{x}^*(t)My(t) + (\Lambda\tilde{x} + \mu)^* N(\Lambda\hat{y} + \lambda) + m\tilde{x} + n(\Lambda\tilde{x} + \mu)\Big]dt + \tilde{x}^*(T)M_T y(T) + m_T\tilde{x}(T)\Big\}.$$

From equality 34 and the above relations find the following relation [3]:

$$E\tilde{x}^*(t)Mk(t) = E\{\tilde{x}^*(t)ME[k(t)|F^T]\} = E\tilde{x}^*(t)M\hat{k}(t). \tag{35}$$

Define,

$$p(t) = \Pi(t)k(t) + r(t), \tag{36}$$

and use it for expression X,

$$X = \int_0^T \Big(\tilde{x}^* My + (\Lambda\tilde{x} + \mu)^* N(\Lambda y + \lambda) + m\tilde{x} + n(\Lambda\tilde{x} + \mu)\Big)dt + \tilde{x}^*(T)p(T). \tag{37}$$

Then the differential form of $\tilde{x}(t)p(t)$ is produced as follows:

$$\begin{aligned} \tfrac{d}{dt}\tilde{x}^* p = \Bigg[&\tilde{x}^*(F^* + \Lambda^* G^*)p + \mu^* G^* p\tilde{x}^* \\ &+ \Big[\tfrac{d\Pi}{dt}k + \Pi(F + G\Lambda)k + \Pi(f + G\lambda) + \tfrac{dr}{dt}\Big]\Bigg]dt \\ &+ \textstyle\sum_{i=1}^d \Big[\tilde{x}^*(B_i^* + \Lambda^* D_i^*)p + \mu^* D_i^* p + \tilde{x}^*\Pi\Big[(B_i + D_i\Lambda)k + D_i\lambda + g_i\Big]\Big]dw_i \\ &+ \textstyle\sum_{i=1}^d \Big[\tilde{x}^*(B_i^* + \Lambda^* D_i^*) + \mu^* D_i^*\Big]\Pi\Big[(B_i + D_i\Lambda)k + D_i\lambda + g_i\Big] + \tilde{x}^*\Pi dw. \end{aligned} \tag{38}$$

By substituting the above relation in 37 find the below expression:

$$\begin{aligned} X = E\textstyle\int_0^T \Big\{&\tilde{x}^* Mk + (\Lambda\tilde{x} + \mu)^* N(\Lambda k + \lambda) + m\tilde{x} + n(\Lambda\tilde{x} + \mu) \\ &+ \tilde{x}^*(F^* + \Lambda^* G^*)(\Pi k + r) + \mu^* G^*(\Pi k + r) + \tilde{x}^*\Big(\tfrac{d\Pi}{dt}k + \tfrac{dr}{dt}\Big) \\ &+ \tilde{x}^*\Pi(F + G\Lambda)k + \tilde{x}^*\Pi(f + G\lambda) \\ &+ \textstyle\sum_{i=1}^d \Big[\tilde{x}^*(B_i^* + \Lambda^* D_i^*) + \mu^* D_i^*\Big]\Pi\Big[(B_i + D_i\Lambda)k + D_i\lambda + g_i\Big]\Big\}dt. \end{aligned} \tag{39}$$

Set the coefficients of $\tilde{x}^*$ and μ^*, zero. This yields the bellow relations [3]:

$$\begin{cases} Mk + \Lambda^* N(\Lambda k + \lambda) + m^* + \Lambda^* n^* + (F^* + \Lambda^* G^*)(\Pi k + r) + \frac{d\Pi}{dt}k + \frac{dr}{dt} + \Pi(F + G\Lambda)k + (f + G\lambda) \\ + \sum_{i=1}^d (B_i^* + \Lambda^* D_i^*)\Pi\Big[(B_i + D_i\Lambda)k + D_i\lambda + g_i\Big] = 0, \end{cases} \tag{40}$$

$$N(\Lambda k + \lambda) + n^* + G^*(\Pi k + r) + \sum_{i=1}^d D_i^*\Pi\Big[(B_i + D_i\Lambda)k + D_i\lambda + g_i\Big] = 0. \tag{41}$$

By matching the coefficients of y and constant terms of equalities 40 and 41, get the following relations:

$$M + \Lambda^* N\Lambda + (F^* + \Lambda^* G^*)\Pi + \tfrac{d\Pi}{dt} + \Pi(F + G\Lambda) + \textstyle\sum_{i=1}^d (B_i^* + \Lambda^* D_i^*)\Pi(B_i + D_i\Lambda) = 0, \tag{42}$$

$$\Lambda^* N\lambda + m^* + (F^* + G\lambda)r + \frac{dr}{dt} + \Pi(f + G\lambda) + \sum_{i=1}^{d}(B_i^* + \Lambda^* D_i^*)\Pi(D_i\lambda + g_i) = 0, \tag{43}$$

$$N\Lambda + G^*\Pi + \sum_{i=1}^{d} D_i^*\Pi(B_i + D_i\Lambda) = 0, \tag{44}$$

$$N\lambda + n^* + G^* r + \sum_{i=1}^{d} D_i^*\Pi(D_i\lambda + g_i) = 0. \tag{45}$$

The following results are achieved by solving the equations 37 and 38 respect to the parameters Λ and λ,

$$\Lambda = -\Big(N + \sum_{i=1}^{d} D_i^*\Pi D_i\Big)^{-1}\Big(G^*\Pi + \sum_{i=1}^{d} D_i^*\Pi B_i\Big), \tag{46}$$

$$\lambda = -\Big(N + \sum_{i=1}^{d} D_i^*\Pi D_i\Big)^{-1}\Big(n^* + G^* r + \sum_{i=1}^{d} D_i^*\Pi g_i\Big). \tag{47}$$

By substituting the above relations in 42 and 43 the formulas 21 and 22 are obtained. Now by solving the equations 21 and 22, find the parameters of relations 40 and 41 for optimal feedback control 9. Therefore, conclude that $X = 0$. It is clear that the optimal process $\mu(\cdot)$ is zero for minimizing the cost functional $J(\nu(\cdot))$. Therefore, it is possible to know that the change of control function is equal to the feedback control. It means that, the control feedback is optimal and unique. □

After finding the optimal feedback control, it should be substituted in the state process. Now there are two dynamic systems so that the second is according to the first. In the next section we introduce a method to simulate this partially observation process.

2.2. *Partially observation systems*

At first, consider the dynamic systems 1 and convert it into the discrete time mode as follows [5]:

$$\begin{cases} x_{k+1} = F_k x_k + f_k + \sum_{i=1}^{d}(B_{i,k}x_k + g_{i,k})w_{i,k} + \sigma_{0,k}w_k, & k = 0, \cdots, N-1, \\ x_0 = \zeta, \\ y_k = H_k x_k + h_k + \sigma_{1,k}b_k, \quad k = 0, \cdots, N-1. \end{cases} \tag{48}$$

The details of this system, has already expressed in the description of system 1. Let us consider the sequence of $\sigma - algebras$,

$$y^k = \sigma(y_0, \ldots, y_{k-1}), \quad k = 1, \ldots, N. \tag{49}$$

The best solution of the partially observable system 48 is to compute the following relation [8]:

$$\hat{x}_N = E[x_N | y^N].$$

In fact the solution of the discrete system 48 is according to the observation process until time N. In this way, use the linear filters. A best linear filter is defined as follows [3]:

$$F_s = \bar{x}_N + \sum_{k=0}^{N-1} S_k(y_k - \bar{y}_k), \tag{50}$$

where $\{S_0, \dots, S_{N-1}\}$ is a set of matrices which characterized by S in the text. The processes $\bar{x}_N$ and $\bar{y}_k$ represent the means of x_N and y_k respectively. The sequences $\bar{x}_k$ and $\bar{y}_k$ are defined as follows:

$$\begin{cases} \bar{x}_{k+1} = F_k \bar{x}_k + f_k, & k = 0, \dots, N-1, \\ \bar{y}_k = H_k \bar{x}_k + h_k, & k = 0, \dots, N-1. \end{cases} \tag{51}$$

The best linear filter is obtained by choosing S in order to minimize the bellow functional:

$$L(S) = E(x_N - F_s)^*(x_N - F_s). \tag{52}$$

Let Λ_{kl} be the correlation matrix of the process x_k , which it is in $L(R^n; R^n)$.

$$\Lambda_{kl} = E(x_k - \bar{x}_k)(x_l - \bar{x}_l)^*. \tag{53}$$

It should be expressed that, $\Lambda^*_{kl} = \Lambda_{lk}$ and if $M \in L(R^n; R^n)$, the following relation is held:

$$E(x_k - \bar{x}_k)^* M(x_l - \bar{x}_l) = \mathrm{tr}\Lambda_{lk} M. \tag{54}$$

Then by using the expression 52, 53 and 54 deduce that,

$$L(S) = E\Big((x_N - \bar{x}_N)(x_N - \bar{x}_N)^* + \Big(\textstyle\sum_{k=0}^{N-1} S_k(y_k - \bar{y}_k)\Big)^2 - 2\Big(\textstyle\sum_{k=0}^{N-1} S_k(y_k - \bar{y}_k)\Big)(x_N - \bar{x}_N)^*\Big),$$

where,

$$E\Big(\textstyle\sum_{k=0}^{N-1} S_k(y_k - \bar{y}_k)\Big)^2 = E\Big(\textstyle\sum_{k,l=0}^{N-1} S_l(y_l - \bar{y}_l) S^*_k (y_k - \bar{y}_k)^*\Big) = \textstyle\sum_{k,l=0}^{N-1} \mathrm{tr}\Lambda_{lk} H^*_k S^*_k S_l H_l + \textstyle\sum_{k=0}^{N-1} \mathrm{tr} R_k S^*_k S_k,$$

and,

$$E\Big(\Big(\textstyle\sum_{k=0}^{N-1} S_k(y_k - \bar{y}_k)\Big)(x_N - \bar{x}_N)^*\Big) = \textstyle\sum_{k=0}^{N-1} E\Big(S_k(y_k - \bar{y}_k)(x_N - \bar{x}_N)^*\Big) = \textstyle\sum_{k=0}^{N-1} \mathrm{tr}\Lambda_{kN} S_k H_k.$$

Therefore,

$$L(S) = \mathrm{tr}\Lambda_{NN} + \sum_{k=0}^{N-1} \mathrm{tr} R_k S^*_k S_k + \sum_{k,l=0}^{N-1} \mathrm{tr}\Lambda_{lk} H^*_k S^*_k S_l H_l - 2\sum_{k=0}^{N-1} \mathrm{tr}\Lambda_{kN} S_k H_k. \tag{55}$$

But this method is usable when S is obtained uniquely.

Theorem 2.7. *There exists a unique S to minimize the functional $L(S)$.*

Proof. See in [3].
□

Now the unique linear filter 50 is existed, but the set S should be found as follows:

$$\frac{d}{dS_i}L(S_0, S_1, \dots, S_{N-1}) = 0, \quad \forall i = 0, 1, \dots, N-1. \tag{56}$$

The above expression is a convenient rule in multiple cases to obtain the minimum or maximum of a function [22]. It is clear that, $L(S)$ is quadratic and positive, so make sure that by using the relation 56 the minimum of $L(S)$ is obtained.

3. Estimation methods

Three methods have been studied in this section to evaluate the problem of the stochastic optimal control with a control factor on the diffusion term. Consider the following optimization problem:

$$\min_{\nu\in U}\Big\{J(\nu(\cdot)) = E\Big[\int_0^T (x^*Mx + \nu^*N\nu + 2mx + 2n\nu)dt + x^*(T)M_Tx(T) + 2m_Tx(T)\Big]\Big\}.$$

$$\begin{cases} dx = \Big(F(t)x + G(t)\nu(t) + f(t)\Big)dt + \sum_{i=1}^d \Big(B_i(t)x + D_i(t)\nu(t) + g_i(t)\Big)dw_i(t) + \sigma_0 dw(t), \\ x(0) = \zeta, \\ dy = \Big[H(t)x + h(t)\Big]dt + \sigma_1 db(t), \\ y(0) = 0. \end{cases}$$

Where, U is the space of admissible controls. The feedback control is obtained by solving differential equation 21 and 22 and substituting them in the control parameters 19 and 20. Now the feedback control can be achieved. Substitute the feedback control in the state process and obtain the following dynamic system:

$$\begin{cases} dx = \Big[F(t)x + G(t)[-N^{-1}G^*\Pi x - N^{-1}n^* - N^{-1}G^*r] + f(t)\Big] + \sigma_0 dw(t) \\ \quad + \sum_{i=1}^d \Big[B_i(t)x + D_i(t)[-N^{-1}G^*\Pi x - N^{-1}n^* - N^{-1}G^*r] + g_i(t)\Big]dw_i(t), \\ x(0) = \zeta, \\ dy = \Big[H(t)x + h(t)\Big]dt + \sigma_1 db(t), \\ y(0) = 0. \end{cases} \tag{57}$$

To estimate this kind of problems, we should use only one step methods. Due to this issue, we have just a terminal or initial values of the processes.

3.1. *Linear filtering method*

This method is to make the discrete system 57 and substitute the solution of 21 and 22 point wise in it. This system is reforming as follows:

$$\begin{cases} x_{k+1} = F_kx_k + G_k[-N_k^{-1}G_k^*\Pi(t_k)x_k - N_k^{-1}n_k^* - N_k^{-1}G_k^*r(t_k)] + f_k + \sigma_{0,k}w_k \\ \quad + \sum_{i=1}^d \Big[B_{i,k}x_k + D_{i,k}[-N_k^{-1}G_k^*\Pi(t_k)x_k - N_k^{-1}n_k^* - N_k^{-1}G_k^*r(t_k)] + g_{i,k}\Big]w_{i,k}, \\ x_0 = \zeta, \\ y_k = H_kx_k + h_k + \sigma_{1,k}, \end{cases} \tag{58}$$

Now, by using the relation 50 and section 2.2 simulat the above system.

In the following subsections we use the simplest numerical methods to evaluate problems, because estimating the stochastic problems are just by the reason of future prediction. It should be mentioned that, using a more accurate method just increases the computational processes.

3.2. Adams-Moulton method

This method is used when the solutions of 21 and 22 do not exist or the state process is non-linear. The Adams method for $y' = f(x, y)$ is $y_{n+1} - y_n = \frac{h}{2}(f_{n+1} + f_n)$. Now, construct 21 and 22 as follows:

$$\begin{cases} \frac{d\Pi}{dt} = -F^*\Pi - \Pi F - \sum_{i=1}^{d} B_i^*\Pi B_i + M + \underbrace{\Big(\Pi G + \sum_{i=1}^{d} B_i^*\Pi D_i\Big)\Big(N + \sum_{i=1}^{d} D_i^*\Pi D_i\Big)^{-1}\Big(G^*\Pi + \sum_{i=1}^{d} D_i^*\Pi B_i\Big)}_{A_1}, \\ \Pi(T) = M_T. \end{cases} \tag{59}$$

$$\begin{cases} \frac{dr}{dt} = -\Big[F^* - \Big(\Pi G + \sum_{i=1}^{d} B_i^*\Pi D_i\Big)\Big(N + \sum_{i=1}^{d} D_i^*\Pi D_i\Big)^{-1} G^*\Big]r - m^* - \Pi f - \sum_{i=1}^{d} B_i^*\Pi g_i \\ \quad + \underbrace{\Big(\Pi G + \sum_{i=1}^{d} B_i^*\Pi D_i\Big)\Big(N + \sum_{i=1}^{d} D_i^*\Pi D_i\Big)^{-1}\Big(n^* + \sum_{i=1}^{d} D_i^*\Pi g_i\Big)}_{A_2}, \\ r(T) = m_T^*. \end{cases} \tag{60}$$

By applying the Adams method on the above equations (equations 59 and 60), two solutions for the equation 59 and four solutions for the equation 60 are found. These solutions are due to the existence of A_1 and A_2. To simulate partially observable system 57, we need unique solutions of the equations 59 and 60 which faster converges to the real solution. So ,using this method (choosing the best answer) is time consuming for this kind of partially observable problem. Therefor, we should use another method to simulate this kind of problem. It should be noted that this method can be used with any solutions of equations 59 and 60. But we are looking for a simpler method that takes less time to simulate. Euler method which is expressed in the bellow is beter than the Adams-Moulton method.

3.3. Euler Method

This method is also used when the solutions 21 and 22 do not exist or the state process is non-linear. As you can see in the previous sections, the dynamics of Π and r do not have an initial value. These dynamics have terminal points $\Pi(T)$ and $r(T)$. So the backward Euler method is used to evaluate the partially observable system. The backward Euler method for $y' = f(x, y)$ is as follows:

$$y_n = y_{n+1} - hf(x_n, y_{n+1}). \tag{61}$$

By applying 61 in the equations 59 and 60 the following results for $k = 1, \ldots, N-1$ are achieved:

$$\begin{cases} \Pi_k = \Pi_{k+1} - h\Big\{ - F_k^*\Pi_{k+1} - \Pi_{k+1}F_k - \sum_{i=1}^{d} B_{i,k}^*\Pi_{k+1}B_{i,k} + M_k \\ \quad + \Big(\Pi_{k+1}G_k + \sum_{i=1}^{d} B_{i,k}^*\Pi_{k+1}D_{i,k}\Big)\Big(N_k + \sum_{i=1}^{d} D_{i,k}^*\Pi_{k+1}D_{i,k}\Big)^{-1}\Big(G_k^*\Pi_{k+1} + \sum_{i=1}^{d} D_{i,k}^*\Pi_{k+1}B_{i,k}\Big)\Big\}. \\ \Pi_N = M_T. \end{cases} \tag{62}$$

$$\begin{cases} r_k = r_{k+1} - h\Big\{ -\Big[F_k^* - \Big(\Pi_k G_k + \sum_{i=1}^d B_{i,k}^* \Pi_k D_{i,k}\Big) \Big(N_k + \sum_{i=1}^d D_{i,k}^* \Pi_k D_{i,k}\Big)^{-1} G_k^*\Big] r_{k+1} - m_k^* - \Pi_k f_k \\ \quad - \sum_{i=1}^d B_{i,k}^* \Pi_k g_{i,k} + \Big(\Pi_k G_k + \sum_{i=1}^d B_{i,k}^* \Pi_k D_{i,k}\Big)\Big(N_k + \sum_{i=1}^d D_{i,k}^* \Pi_k D_{i,k}\Big)^{-1}\Big(n_k^* + \sum_{i=1}^d D_{i,k}^* \Pi_k g_{i,k}\Big)\Big\}. \\ r_N = m_T^*. \end{cases} \tag{63}$$

Now the solutions of r_k and Π_k for $k = 1, \ldots, N-1$ are available. By substituting them in the discrete form of system 57, the following expressions are achieved:

$$\begin{cases} x_{k+1} = F_k x_k + G_k[-N_k^{-1} G_k^* \Pi_k x_k - N_k^{-1} n_k^* - N_k^{-1} G_k^* r_k] + f_k + \sigma_{0,k} w_k \\ \qquad + \sum_{i=1}^d \Big[B_{i,k} x_k + D_{i,k}[-N_k^{-1} G_k^* \Pi_k x_k - N_k^{-1} n_k^* - N_k^{-1} G_k^* r_k] + g_{i,k}\Big] w_{i,k}, \\ x_0 = \zeta, \\ y_k = H_k x_k + h_k + \sigma_{1,k}. \end{cases} \tag{64}$$

So, by using the relation 50 and section 2.2, simulate the above system.

4. Numerical Example

In this section, two examples are provided. These examples are in linear and nonlinear cases respectively. Example 4.1 is linear and is solved by the 'linear filter' method and Euler method. But example 4.2 is nonlinear and is just solved by Euler method.

Example 4.1. *Consider the following stochastic optimal control with the partial observation system,*

$$min_{u\in U}\Big\{E\Big[\int_0^1 (u^2 - 2u)dt + x^2(1)\Big]\Big\}, \tag{65}$$

$$dx = (x+1)dt + (u+1)dw_i + dw, \quad x_0 = 0.5. \tag{66}$$

$$dy = (x+1)dt + db, \quad y_0 = 0. \tag{67}$$

This linear example is solved by the 'linear filter' method and Euler method. The control factor of this example is as follows:

$$u(t) = \frac{1}{1 + \exp(-2t + 2)}.$$

In figure (1)*, it has prepared a simulation of the corresponding partial observation system by Linear filter method.*

Figure 1: Five paths of linear filtering method

Once again we simulate the current example by using Backward Euler method; the results are presented in figure (2).

Figure 2: Five paths by Euler method

Finally we compare the two previous results according to the numerical aspects in figure 3.

Figure 3: Comparison of Two Methods

Example 4.2. *Consider the following nonlinear problem,*

$$min_{u\in U}\Big\{E\Big[\int_0^1 (u^2+2u)dt + x^2(1)\Big]\Big\}, \tag{68}$$

$$dx = \Big(sin(\frac{\pi t}{2})x+1\Big)dt + (u+1)dw_i + dw, \quad x_0 = 0.5. \tag{69}$$

$$dy = (x+1)dt + db, \quad y_0 = 0. \tag{70}$$

The state process in this example is not linear and the equations 21 and 22 are not solvable for this problem. So, we should use the Euler method for estimation.
In this example the control factor is as follows:

$$u(t) = -1.$$

We simulate the current example by using Backward Euler method and the results are presented in figure 4.

Figure 4: Five paths by Euler method

5. *Conclusion*

This paper studies SOCPPs with the 'control factor' on the diffusion term. First it introduces the linear filtering method to estimat this kind of problem. But this method can be used when the dynamics of control parameters have analytical solutions. The second and third methods are used when these solutions do not exist. But, the solution of the second method is not unique. So, it is time consuming to be used in calculations. Instead, the third method, the weak solution of the problem, can be used. According to the example 4.1, the first and the third methods are numerically the same .

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