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Some New Chebyshev and Grüss-type Integral Inequalities for Saigo Fractional Integral Operators and Their q-analogues

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Abstract. By making use of Saigo fractional integral operators, we establish some new results of the fractional Chebyshev and Grüss-type integral inequalities. Furthermore, the q-extensions of the main results are also presented. Our results in special cases yield some of the recent results on Chebyshev and Grüss-type integral inequalities.

1. Introduction

Recently, by applying the fractional integral operators and the fractional q-integral operators, many researchers have obtained a lot of fractional integral inequalities and fractional q-integral inequalities and applications. For example, we refer the reader to [3–5, 7, 28, 30] and the references cited therein. Belarbi and Dahmani [6] gave the following integral inequality, using the Riemann-Liouville fractional integrals: if f and g are two synchronous functions on $C[0,\infty)$, then

$$R^{\alpha}(fg)(t) \geq \frac{\Gamma(\alpha+1)}{t^{\alpha}} R^{\alpha}f(t)R^{\alpha}g(t), \tag{1}$$

and

$$\frac{t^{\alpha}}{\Gamma(\alpha+1)}R^{\beta}(fg)(t) + \frac{t^{\beta}}{\Gamma(\beta+1)}R^{\alpha}(fg)(t) \geq R^{\alpha}f(t)R^{\beta}g(t) + R^{\beta}f(t)R^{\alpha}g(t), \tag{2}$$

for all $t > 0$, $\alpha > 0$, and $\beta > 0$. Öğünmez and Özkan [24], Chinchane and Pachpatte [11] and Purohit and Raina [25] obtained the Riemann-Liouville fractional q-integral inequalities, the Hadamard fractional integral inequalities and the Saigo fractional integral and q-integral inequalities similar to the inequalities (1) and (2), respectively. Here we should point out that the Saigo fractional integral and q-integral inequalities include the Riemann-Liouville fractional integral and q-integral inequalities, respectively.

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Dahmani [13] established the following fractional integral inequalities which are generalizations of the inequalities (1) and (2), by using the Riemann-Liouville fractional integrals. Let f and g be two synchronous functions on $[0,\infty)$ and let $u, v : [0,\infty) \to [0,\infty)$. Then

$$R^{\alpha}u(t)R^{\alpha}(vfg)(t) + R^{\alpha}v(t)R^{\alpha}(ufg)(t) \geq R^{\alpha}(uf)(t)R^{\alpha}(vg)(t) + R^{\alpha}(vf)(t)R^{\alpha}(ug)(t). \tag{3}$$

and

$$R^{\alpha}u(t)R^{\beta}(vfg)(t) + R^{\beta}v(t)R^{\alpha}(ufg)(t) \geq R^{\alpha}(uf)(t)R^{\beta}(vg)(t) + R^{\beta}(vf)(t)R^{\alpha}(ug)(t). \tag{4}$$

for all $t > 0$, $\alpha > 0$ and $\beta > 0$. Yang [31], Brahim and Taf [9] and Chinchane and Pachpatte [12] gave the fractional q-integral inequalities, the fractional integral inequalities with two parameters of deformation q_1 and q_2, and the Hadamard fractional integral inequalities similar to inequalities (3) and (4), respectively.

Let us consider the celebrated Chebyshev functional (see [10, 22])

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx \cdot \frac{1}{b-a}\int_a^b g(x)dx \tag{5}$$

where f and g are two integrable functions on $[a,b]$. In [23], Grüss proved the well known inequality:

$$|T(f,g)| \leq \frac{1}{4}(\Phi-\phi)(\Psi-\psi), \tag{6}$$

where f and g are two integrable functions on $[a,b]$ satisfying the conditions

$$\phi \leq f(x) \leq \Phi, \quad \psi \leq g(x) \leq \Psi, \quad \phi,\Phi,\psi,\Psi \in \mathbb{R}, \ \ x \in [a,b]. \tag{7}$$

The inequality (6) is known Grüss' inequality. It has gained a considerable attentions, for example, under suitable assumptions for the involved operators, Dragomir [18] gave some inequalities of Grüss' type for vectors and continuous functions of selfadjoint operators in Hilbert spaces. In the case of f, g satisfying the conditions (7), Dragomir (see [19]) proved that

$$|S(f,g,p)| \leq \frac{1}{4}(\Phi-\phi)(\Psi-\psi)\Big(\int_a^b p(x)dx\Big)^2, \tag{8}$$

where

$$S(f,g,p) = \frac{1}{2}T(f,g,p,p) = \int_a^b p(x)dx\int_a^b p(x)f(x)g(x)dx - \int_a^b p(x)f(x)dx\int_a^b p(x)g(x)dx, \tag{9}$$

and

$$\begin{aligned} T(f,g,p,q) = {} & \int_a^b q(x)dx\int_a^b p(x)f(x)g(x)dx + \int_a^b p(x)dx\int_a^b q(x)f(x)g(x)dx \\ & - \int_a^b q(x)f(x)dx\int_a^b p(x)g(x)dx - \int_a^b p(x)f(x)dx\int_a^b q(x)g(x)dx. \end{aligned} \tag{10}$$

In the case of $f', g' \in L_{\infty}(a,b)$, Dragomir (see [19]) proved that

$$|S(f,g,p)| \leq \|f'\|_{\infty}\|g'\|_{\infty}\Big(\int_a^b p(x)dx\int_a^b x^2p(x)dx - \Big(\int_a^b xp(x)dx\Big)^2\Big). \tag{11}$$

If f is M-g-Lipschitzian on $[a,b]$, i.e.,

$$|f(x)-f(y)| \le M|g(x)-g(y)|, \quad M>0, \ \ x,y\in[a,b], \tag{12}$$

Dragomir (see [19]) proved that

$$|S(f,g,p)| \le M\Big(\int_a^b p(x)dx\int_a^b g^2(x)p(x)dx-\Big(\int_a^b g(x)p(x)dx\Big)^2\Big). \tag{13}$$

If f is an L_1-lipschitzian function on $[a,b]$ and g is an L_2-lipschitzian function on $[a,b]$, Dragomir (see [19]) proved that

$$|S(f,g,p)| \le L_1L_2\Big(\int_a^b p(x)dx\int_a^b x^2p(x)dx-\Big(\int_a^b xp(x)dx\Big)^2\Big). \tag{14}$$

By using the Riemann-Liouville fractional integral and q-integral operators, Dahmani et al. [16] and Zhu et al. [32] gave the fractional integral and q-integral inequality similar to inequality (6) satisfying the conditions (7), respectively. By using the Riemann-Liouville fractional q-integral operators, Dahmani and Benzidane [15] gave the fractional q-integral inequality similar to (8) satisfying the conditions (5). Dahmani [14] and Dahmani et al [17] obtained the fractional integral inequalities for the extended functional (10) similar to inequalities (11), (13) and (14), using the Riemann-Liouville fractional integrals. By using the Riemann-Liouville fractional q-integral operators, Brahim and Taf [8, 9] established the fractional q-integral inequalities and fractional integral inequalities for the extended functional (10) with two parameters of deformation q_1 and q_2 similar to inequalities (11), (13) and (14), respectively.

Motivated by the above mentioned works, the main aim of this paper is to establish some new results of the fractional Chebyshev and Grüss-type integral inequalities involving the Saigo fractional integral operators. Furthermore, the q-extensions of the main results are also considered. Our results obviously include the recent results on Chebyshev and Grüss-type integral inequalities.

2. Saigo Fractional Integral Inequalities

Before stating the Saigo fractional integral inequalities, we mention below the definitions and notations of some well-known operators of fractional calculus , we can see [21, 25].

Definition 2.1 ([21, 25]). A real-valued function $f(t)$ $(t>0)$ is said to be in the space $\mathbb{C}_\mu(\mu\in\mathbb{R})$, if there exists a real number $p>\mu$ such that $f(t)=t^p\phi(t)$, where $\phi(t)\in\mathbb{C}(0,\infty)$.

Definition 2.2 ([21, 25]). Let $\alpha>0,\beta,\eta\in\mathbb{R}$, then the Saigo fractional integral $I_{0,t}^{\alpha,\beta,\eta}$ of order α for a real-valued continuous function $f(t)$ is defined by

$$I_{0,t}^{\alpha,\beta,\eta}f(t)=\frac{t^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}{}_2F_1\Big(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\Big)f(\tau)d\tau, \tag{15}$$

where, the function ${}_2F_1(-)$ in the right-hand side of (15) is the Gaussian hypergeometric function defined by

$${}_2F_1(a,b;c;t)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{t^n}{n!}, \tag{16}$$

and $(a)_n$ is the Pochhammer symbol $(a)_n=a(a+1)\cdots(a+n-1), \quad (a)_0=1.$

The integral operator $I_{0,t}^{\alpha,\beta,\eta}$ includes both the Riemann-Liouville and the Erdélyi-Kober fractional integral operators given by the following relationships:

$$R^{\alpha}f(t) = I_{0,t}^{\alpha,-\alpha,\eta}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau)d\tau, \quad (\alpha > 0) \tag{17}$$

and

$$I^{\alpha,\eta}f(t) = I_{0,t}^{\alpha,0,\eta}f(t) = \frac{t^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\tau^{\eta}f(\tau)d\tau, \quad (\alpha > 0,\ \eta \in \mathbb{R}). \tag{18}$$

For $f(t) = t^{\mu}$ in (15), we get the known formula:

$$I_{0,t}^{\alpha,\beta,\eta}t^{\mu} = \frac{\Gamma(\mu+1)\Gamma(\mu+1-\beta+\eta)}{\Gamma(\mu+1-\beta)\Gamma(\mu+1+\alpha+\eta)}t^{\mu-\beta}, \tag{19}$$

for all $t > 0$, $\alpha > 0$, $\min(\mu, \mu-\beta+\eta) > -1$.

In this section, we firstly give some new Chebyshev-type integral inequalities for the synchronous functions involving the Saigo fractional integral operators.

Lemma 2.3. *Let f and g be two synchronous functions on $[0,\infty)$ and let u and v be two nonnegative continuous functions on $[0,\infty)$. Then we have*

$$I_{0,t}^{\alpha,\beta,\eta}u(t)I_{0,t}^{\alpha,\beta,\eta}(vfg)(t) + I_{0,t}^{\alpha,\beta,\eta}v(t)I_{0,t}^{\alpha,\beta,\eta}(ufg)(t) \geq I_{0,t}^{\alpha,\beta,\eta}(uf)(t)I_{0,t}^{\alpha,\beta,\eta}(vg)(t) + I_{0,t}^{\alpha,\beta,\eta}(vf)(t)I_{0,t}^{\alpha,\beta,\eta}(ug)(t), \tag{20}$$

for all $t > 0$, $\alpha > \max\{0, -\beta\}$, $\beta < 1$, $\beta - 1 < \eta < 0$.

Proof. Consider

$$\begin{aligned} F(t,\tau) &= \frac{t^{-\alpha-\beta}(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}{}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{\tau}{t}\right) \quad (\tau \in (0,t);\ t > 0) \\ &= \frac{1}{\Gamma(\alpha)}\frac{(t-\tau)^{\alpha-1}}{t^{\alpha+\beta}} + \frac{(\alpha+\beta)(-\eta)}{\Gamma(\alpha+1)}\frac{(t-\tau)^{\alpha}}{t^{\alpha+\beta+1}} + \frac{(\alpha+\beta)(\alpha+\beta+1)(-\eta)(-\eta+1)}{\Gamma(\alpha+2)}\frac{(t-\tau)^{\alpha+1}}{t^{\alpha+\beta+2}} + \cdots. \end{aligned} \tag{21}$$

We observe that the function $F(t,\tau)$ remains positive, for all $\tau \in (0,t)$ $(t > 0)$ since each term of the above series is positive in view of the conditions stated with Lemma 2.3.

Since f and g are two synchronous functions on $[0,\infty)$, then for all $\tau > 0$ and $\rho > 0$, we have

$$(f(\tau) - f(\rho))(g(\tau) - g(\rho)) \geq 0. \tag{22}$$

By (22), we write

$$f(\tau)g(\tau) + f(\rho)g(\rho) \geq f(\tau)g(\rho) + f(\rho)g(\tau). \tag{23}$$

Multiplying both side of (23) by $v(\tau)F(t,\tau)$ ($F(t,\tau)$ defined by (21)) and integrating the resulting identity with respect to τ from 0 to t, we get

$$I_{0,t}^{\alpha,\beta,\eta}(vfg)(t) + f(\rho)g(\rho)I_{0,t}^{\alpha,\beta,\eta}v(t) \geq g(\rho)I_{0,t}^{\alpha,\beta,\eta}(vf)(t) + f(\rho)I_{0,t}^{\alpha,\beta,\eta}(vg)(t). \tag{24}$$

Multiplying both side of (24) by $u(\rho)F(t,\rho)$ and integrating the resulting identity with respect to ρ from 0 to t, we obtain

$$I_{0,t}^{\alpha,\beta,\eta}u(t)I_{0,t}^{\alpha,\beta,\eta}(vfg)(t) + I_{0,t}^{\alpha,\beta,\eta}v(t)I_{0,t}^{\alpha,\beta,\eta}(ufg)(t) \geq I_{0,t}^{\alpha,\beta,\eta}(vf)(t)I_{0,t}^{\alpha,\beta,\eta}(ug)(t) + I_{0,t}^{\alpha,\beta,\eta}(uf)(t)I_{0,t}^{\alpha,\beta,\eta}(vg)(t), \tag{25}$$

which implies (20). □

Theorem 2.4. *Let f and g be two synchronous functions on $[0,\infty)$ and let x, y and z be three nonnegative continuous functions on $[0,\infty)$. Then we have*

$$\begin{aligned}
&2I_{0,t}^{\alpha,\beta,\eta}x(t)\Big(I_{0,t}^{\alpha,\beta,\eta}y(t)I_{0,t}^{\alpha,\beta,\eta}(zfg)(t)+I_{0,t}^{\alpha,\beta,\eta}z(t)I_{0,t}^{\alpha,\beta,\eta}(yfg)(t)\Big)+2I_{0,t}^{\alpha,\beta,\eta}y(t)I_{0,t}^{\alpha,\beta,\eta}z(t)I_{0,t}^{\alpha,\beta,\eta}(xfg)(t)\\
&\geq I_{0,t}^{\alpha,\beta,\eta}x(t)\Big(I_{0,t}^{\alpha,\beta,\eta}(yf)(t)I_{0,t}^{\alpha,\beta,\eta}(zg)(t)+I_{0,t}^{\alpha,\beta,\eta}(zf)(t)I_{0,t}^{\alpha,\beta,\eta}(yg)(t)\Big)+I_{0,t}^{\alpha,\beta,\eta}y(t)\Big(I_{0,t}^{\alpha,\beta,\eta}(xf)(t)I_{0,t}^{\alpha,\beta,\eta}(zg)(t)\\
&\quad+I_{0,t}^{\alpha,\beta,\eta}(zf)(t)I_{0,t}^{\alpha,\beta,\eta}(xg)(t)\Big)+I_{0,t}^{\alpha,\beta,\eta}z(t)\Big(I_{0,t}^{\alpha,\beta,\eta}(xf)(t)I_{0,t}^{\alpha,\beta,\eta}(yg)(t)+I_{0,t}^{\alpha,\beta,\eta}(yf)(t)I_{0,t}^{\alpha,\beta,\eta}(xg)(t)\Big), \qquad (26)
\end{aligned}$$

for all $t>0$, $\alpha>\max\{0,-\beta\}$, $\beta<1$, $\beta-1<\eta<0$.

Proof. Putting $u=y$, $v=z$ and using Lemma 2.3, we can write

$$I_{0,t}^{\alpha,\beta,\eta}y(t)I_{0,t}^{\alpha,\beta,\eta}(zfg)(t)+I_{0,t}^{\alpha,\beta,\eta}z(t)I_{0,t}^{\alpha,\beta,\eta}(yfg)(t)\geq I_{0,t}^{\alpha,\beta,\eta}(yf)(t)I_{0,t}^{\alpha,\beta,\eta}(zg)(t)+I_{0,t}^{\alpha,\beta,\eta}(zf)(t)I_{0,t}^{\alpha,\beta,\eta}(yg)(t), \qquad (27)$$

Multiplying both sides of (27) by $I_{0,t}^{\alpha,\beta,\eta}x(t)$, we obtain

$$\begin{aligned}
&I_{0,t}^{\alpha,\beta,\eta}x(t)\Big(I_{0,t}^{\alpha,\beta,\eta}y(t)I_{0,t}^{\alpha,\beta,\eta}(zfg)(t)+I_{0,t}^{\alpha,\beta,\eta}z(t)I_{0,t}^{\alpha,\beta,\eta}(yfg)(t)\Big)\\
&\qquad\geq I_{0,t}^{\alpha,\beta,\eta}x(t)\Big(I_{0,t}^{\alpha,\beta,\eta}(yf)(t)I_{0,t}^{\alpha,\beta,\eta}(zg)(t)+I_{0,t}^{\alpha,\beta,\eta}(zf)(t)I_{0,t}^{\alpha,\beta,\eta}(yg)(t)\Big), \qquad (28)
\end{aligned}$$

Putting $u=x$, $v=z$ and using Lemma 2.3, we can write

$$I_{0,t}^{\alpha,\beta,\eta}x(t)I_{0,t}^{\alpha,\beta,\eta}(zfg)(t)+I_{0,t}^{\alpha,\beta,\eta}z(t)I_{0,t}^{\alpha,\beta,\eta}(xfg)(t)\geq I_{0,t}^{\alpha,\beta,\eta}(xf)(t)I_{0,t}^{\alpha,\beta,\eta}(zg)(t)+I_{0,t}^{\alpha,\beta,\eta}(zf)(t)I_{0,t}^{\alpha,\beta,\eta}(xg)(t), \qquad (29)$$

Multiplying both sides of (29) by $I_{0,t}^{\alpha,\beta,\eta}y(t)$, we obtain

$$\begin{aligned}
&I_{0,t}^{\alpha,\beta,\eta}y(t)\Big(I_{0,t}^{\alpha,\beta,\eta}x(t)I_{0,t}^{\alpha,\beta,\eta}(zfg)(t)+I_{0,t}^{\alpha,\beta,\eta}z(t)I_{0,t}^{\alpha,\beta,\eta}(xfg)(t)\Big)\\
&\qquad\geq I_{0,t}^{\alpha,\beta,\eta}y(t)\Big(I_{0,t}^{\alpha,\beta,\eta}(xf)(t)I_{0,t}^{\alpha,\beta,\eta}(zg)(t)+I_{0,t}^{\alpha,\beta,\eta}(zf)(t)I_{0,t}^{\alpha,\beta,\eta}(xg)(t)\Big), \qquad (30)
\end{aligned}$$

With the same arguments as before, we can get

$$\begin{aligned}
&I_{0,t}^{\alpha,\beta,\eta}z(t)\Big(I_{0,t}^{\alpha,\beta,\eta}x(t)I_{0,t}^{\alpha,\beta,\eta}(yfg)(t)+I_{0,t}^{\alpha,\beta,\eta}y(t)I_{0,t}^{\alpha,\beta,\eta}(xfg)(t)\Big)\\
&\qquad\geq I_{0,t}^{\alpha,\beta,\eta}z(t)\Big(I_{0,t}^{\alpha,\beta,\eta}(xf)(t)I_{0,t}^{\alpha,\beta,\eta}(yg)(t)+I_{0,t}^{\alpha,\beta,\eta}(yf)(t)I_{0,t}^{\alpha,\beta,\eta}(xg)(t)\Big), \qquad (31)
\end{aligned}$$

The required inequality (26) follows on adding the inequalities (28), (30) and (31). □

Lemma 2.5. *Let f and g be two synchronous functions on $[0,\infty)$ and let u and v be two nonnegative continuous functions on $[0,\infty)$. Then we have*

$$I_{0,t}^{\alpha,\beta,\eta}u(t)I_{0,t}^{\gamma,\delta,\zeta}(vfg)(t)+I_{0,t}^{\gamma,\delta,\zeta}v(t)I_{0,t}^{\alpha,\beta,\eta}(ufg)(t)\geq I_{0,t}^{\alpha,\beta,\eta}(uf)(t)I_{0,t}^{\gamma,\delta,\zeta}(vg)(t)+I_{0,t}^{\gamma,\delta,\zeta}(vf)(t)I_{0,t}^{\alpha,\beta,\eta}(ug)(t), \qquad (32)$$

for all $t>0$, $\alpha>\max\{0,-\beta\}$, $\gamma>\max\{0,-\delta\}$, $\beta,\delta<1$, $\beta-1<\eta<0$, $\delta-1<\zeta<0$.

Proof. Multiplying both sides of (23) by $v(\rho)G(t,\rho)$, where

$$G(t,\rho)=\frac{t^{-\gamma-\delta}(t-\rho)^{\gamma-1}}{\Gamma(\gamma)}{}_2F_1\left(\gamma+\delta,-\zeta;\gamma;1-\frac{\rho}{t}\right)\quad(\rho\in(0,t);\ t>0). \tag{33}$$

In view of the arguments mentioned above in the proof of Lemma 2.3. We can see that the function $G(t,\tau)$ remains positive under the conditions stated with Lemma 2.5. Integrating the resulting inequality obtained with respect to ρ from 0 to t, we have

$$f(\tau)g(\tau)I_{0,t}^{\gamma,\delta,\zeta}v(t)+I_{0,t}^{\gamma,\delta,\zeta}(vfg)(t)\geq f(\tau)I_{0,t}^{\gamma,\delta,\zeta}(vg)(t)+g(\tau)I_{0,t}^{\gamma,\delta,\zeta}(vf)(t). \tag{34}$$

Multiplying both side of (34) by $u(\tau)F(t,\tau)$ (defined by (21)) and integrating the resulting identity with respect to τ from 0 to t, we obtain

$$I_{0,t}^{\gamma,\delta,\zeta}v(t)I_{0,t}^{\alpha,\beta,\eta}(ufg)(t)+I_{0,t}^{\alpha,\beta,\eta}u(t)I_{0,t}^{\gamma,\delta,\zeta}(vfg)(t)\geq I_{0,t}^{\alpha,\beta,\eta}(uf)(t)I_{0,t}^{\gamma,\delta,\zeta}(vg)(t)+I_{0,t}^{\gamma,\delta,\zeta}(vf)(t)I_{0,t}^{\alpha,\beta,\eta}(ug)(t), \tag{35}$$

which implies (32). □

Theorem 2.6. *Let f and g be two synchronous functions on $[0,\infty)$ and let x, y and z be three nonnegative continuous functions on $[0,\infty)$. Then we have*

$$\begin{aligned}
&I_{0,t}^{\alpha,\beta,\eta}x(t)\Big(I_{0,t}^{\alpha,\beta,\eta}z(t)I_{0,t}^{\gamma,\delta,\zeta}(yfg)(t)+2I_{0,t}^{\alpha,\beta,\eta}y(t)I_{0,t}^{\gamma,\delta,\zeta}(zfg)(t)+I_{0,t}^{\gamma,\delta,\zeta}z(t)I_{0,t}^{\alpha,\beta,\eta}(yfg)(t)\Big)\\
&+\Big(I_{0,t}^{\alpha,\beta,\eta}y(t)I_{0,t}^{\gamma,\delta,\zeta}z(t)+I_{0,t}^{\gamma,\delta,\zeta}y(t)I_{0,t}^{\alpha,\beta,\eta}z(t)\Big)I_{0,t}^{\alpha,\beta,\eta}(xfg)(t)\\
&\geq I_{0,t}^{\alpha,\beta,\eta}x(t)\Big(I_{0,t}^{\alpha,\beta,\eta}(yf)(t)I_{0,t}^{\gamma,\delta,\zeta}(zg)(t)+I_{0,t}^{\gamma,\delta,\zeta}(zf)(t)I_{0,t}^{\alpha,\beta,\eta}(yg)(t)\Big)+I_{0,t}^{\gamma,\delta,\zeta}y(t)\Big(I_{0,t}^{\alpha,\beta,\eta}(xf)(t)I_{0,t}^{\gamma,\delta,\zeta}(zg)(t)\\
&\qquad+I_{0,t}^{\gamma,\delta,\zeta}(zf)(t)I_{0,t}^{\alpha,\beta,\eta}(xg)(t)\Big)+I_{0,t}^{\alpha,\beta,\eta}z(t)\Big(I_{0,t}^{\alpha,\beta,\eta}(xf)(t)I_{0,t}^{\gamma,\delta,\zeta}(yg)(t)+I_{0,t}^{\gamma,\delta,\zeta}(yf)(t)I_{0,t}^{\alpha,\beta,\eta}(xg)(t)\Big).
\end{aligned} \tag{36}$$

for all $t>0$, $\alpha>\max\{0,-\beta\}$, $\gamma>\max\{0,-\delta\}$, $\beta,\delta<1$, $\beta-1<\eta<0$, $\delta-1<\zeta<0$.

Proof. Putting $u=y$, $v=z$ and using Lemma 2.5, we can write

$$I_{0,t}^{\alpha,\beta,\eta}y(t)I_{0,t}^{\gamma,\delta,\zeta}(zfg)(t)+I_{0,t}^{\gamma,\delta,\zeta}z(t)I_{0,t}^{\alpha,\beta,\eta}(yfg)(t)\geq I_{0,t}^{\alpha,\beta,\eta}(yf)(t)I_{0,t}^{\gamma,\delta,\zeta}(zg)(t)+I_{0,t}^{\gamma,\delta,\zeta}(zf)(t)I_{0,t}^{\alpha,\beta,\eta}(yg)(t), \tag{37}$$

Multiplying both sides of (37) by $I_{0,t}^{\alpha,\beta,\eta}x(t)$, we obtain

$$\begin{aligned}
&I_{0,t}^{\alpha,\beta,\eta}x(t)\Big(I_{0,t}^{\alpha,\beta,\eta}y(t)I_{0,t}^{\gamma,\delta,\zeta}(zfg)(t)+I_{0,t}^{\gamma,\delta,\zeta}z(t)I_{0,t}^{\alpha,\beta,\eta}(yfg)(t)\Big)\\
&\qquad\geq I_{0,t}^{\alpha,\beta,\eta}x(t)\Big(I_{0,t}^{\alpha,\beta,\eta}(yf)(t)I_{0,t}^{\gamma,\delta,\zeta}(zg)(t)+I_{0,t}^{\gamma,\delta,\zeta}(zf)(t)I_{0,t}^{\alpha,\beta,\eta}(yg)(t)\Big),
\end{aligned} \tag{38}$$

Putting $u=x$, $v=z$ and using Lemma 2.5, we can write

$$I_{0,t}^{\alpha,\beta,\eta}x(t)I_{0,t}^{\gamma,\delta,\zeta}(zfg)(t)+I_{0,t}^{\gamma,\delta,\zeta}z(t)I_{0,t}^{\alpha,\beta,\eta}(xfg)(t)\geq I_{0,t}^{\alpha,\beta,\eta}(xf)(t)I_{0,t}^{\gamma,\delta,\zeta}(zg)(t)+I_{0,t}^{\gamma,\delta,\zeta}(zf)(t)I_{0,t}^{\alpha,\beta,\eta}(xg)(t), \tag{39}$$

Multiplying both sides of (39) by $I_{0,t}^{\alpha,\beta,\eta}y(t)$, we obtain

$$\begin{aligned}
&I_{0,t}^{\alpha,\beta,\eta}y(t)\Big(I_{0,t}^{\alpha,\beta,\eta}x(t)I_{0,t}^{\gamma,\delta,\zeta}(zfg)(t)+I_{0,t}^{\gamma,\delta,\zeta}z(t)I_{0,t}^{\alpha,\beta,\eta}(xfg)(t)\Big)\\
&\qquad\geq I_{0,t}^{\alpha,\beta,\eta}y(t)\Big(I_{0,t}^{\alpha,\beta,\eta}(xf)(t)I_{0,t}^{\gamma,\delta,\zeta}(zg)(t)+I_{0,t}^{\gamma,\delta,\zeta}(zf)(t)I_{0,t}^{\alpha,\beta,\eta}(xg)(t)\Big),
\end{aligned} \tag{40}$$

With the same arguments as before, we can get

$$I_{0,t}^{\alpha,\beta,\eta}z(t)\Big(I_{0,t}^{\alpha,\beta,\eta}x(t)I_{0,t}^{\gamma,\delta,\zeta}(yfg)(t)+I_{0,t}^{\gamma,\delta,\zeta}y(t)I_{0,t}^{\alpha,\beta,\eta}(xfg)(t)\Big)$$
$$\geq I_{0,t}^{\alpha,\beta,\eta}z(t)\Big(I_{0,t}^{\alpha,\beta,\eta}(xf)(t)I_{0,t}^{\gamma,\delta,\zeta}(yg)(t)+I_{0,t}^{\gamma,\delta,\zeta}(yf)(t)I_{0,t}^{\alpha,\beta,\eta}(xg)(t)\Big), \quad (41)$$

The required inequality (36) follows on adding the inequalities (38), (40) and (41). □

Remark 2.7. The inequalities (26) and (36) are reversed in the following cases: (*a*) The functions f and g asynchronous on $[0,\infty)$. (*b*) The functions x, y and z are negative on $[0,\infty)$. (*c*) Two of he functions x, y and z are positive and the third one is negative on $[0,\infty)$.

Remark 2.8. For $\alpha=\gamma$, $\beta=\delta$, $\eta=\zeta$, Lemma 2.5 and Theorem 2.6 immediately reduce to Lemma 2.3 and Theorem 2.4, respectively. For $u(t)=v(t)=1$, Lemmas 2.3 and 2.5 immediately reduce to Theorems 1 and 2 in [25], respectively. If we replace β by $-\alpha$ (and δ by $-\gamma$ additionally for Lemma 2.5 and Theorem 2.6), and make use of the relation (17), then Lemmas 2.3 and 2.5 and Theorems 2.4 and 2.6 correspond to the known results due to Dahmani [14]. Furthermore, set $u(t)=v(t)=1$, then Lemmas 2.3 and 2.5 immediately reduce to the known results due to Belarbi and Dahmani [6].

By putting $\beta=0$ (and $\delta=0$ additionally for Lemma 2.5 and Theorem 2.6), and using the relation (18), Lemmas 2.3 and 2.5 and Theorem 2.4 and 2.6 yield the following fractional integral inequalities involving the Erdélyi-Kober type fractional integral operators defined by (18).

Corollary 2.9. *Let f and g be two synchronous functions on $[0,\infty)$ and let u and v be two nonnegative continuous functions on $[0,\infty)$. Then we have*

$$I^{\alpha,\eta}u(t)I^{\alpha,\eta}(vfg)(t)+I^{\alpha,\eta}v(t)I^{\alpha,\eta}(ufg)(t)\geq I^{\alpha,\eta}(uf)(t)I^{\alpha,\eta}(vg)(t)+I^{\alpha,\eta}(vf)(t)I^{\alpha,\eta}(ug)(t), \quad (42)$$

for all $t>0$, $\alpha>0$, $-1<\eta<0$.

Corollary 2.10. *Let f and g be two synchronous functions on $[0,\infty)$ and let x, y and z be three nonnegative continuous functions on $[0,\infty)$. Then we have*

$$2I^{\alpha,\eta}x(t)\Big(I^{\alpha,\eta}y(t)I^{\alpha,\eta}(zfg)(t)+I^{\alpha,\eta}z(t)I^{\alpha,\eta}(yfg)(t)\Big)+2I^{\alpha,\eta}y(t)I^{\alpha,\eta}z(t)I^{\alpha,\eta}(xfg)(t)$$
$$\geq I^{\alpha,\eta}x(t)\Big(I^{\alpha,\eta}(yf)(t)I^{\alpha,\eta}(zg)(t)+I^{\alpha,\eta}(zf)(t)I^{\alpha,\eta}(yg)(t)\Big)+I^{\alpha,\eta}y(t)\Big(I^{\alpha,\eta}(xf)(t)I^{\alpha,\eta}(zg)(t)$$
$$+I^{\alpha,\eta}(zf)(t)I^{\alpha,\eta}(xg)(t)\Big)+I^{\alpha,\eta}z(t)\Big(I^{\alpha,\eta}(xf)(t)I^{\alpha,\eta}(yg)(t)+I^{\alpha,\eta}(yf)(t)I^{\alpha,\eta}(xg)(t)\Big), \quad (43)$$

for all $t>0$, $\alpha>0$, $-1<\eta<0$.

Corollary 2.11. *Let f and g be two synchronous functions on $[0,\infty)$ and let u and v be two nonnegative continuous functions on $[0,\infty)$. Then we have*

$$I^{\alpha,\eta}u(t)I^{\gamma,\zeta}(vfg)(t)+I^{\gamma,\zeta}v(t)I^{\alpha,\eta}(ufg)(t)\geq I^{\alpha,\eta}(uf)(t)I^{\gamma,\zeta}(vg)(t)+I^{\gamma,\zeta}(vf)(t)I^{\alpha,\eta}(ug)(t), \quad (44)$$

for all $t>0$, $\alpha,\gamma>0$, $-1<\max(\eta,\zeta)<0$.

Corollary 2.12. *Let f and g be two synchronous functions on $[0,\infty)$ and let x, y and z be three nonnegative continuous functions on $[0,\infty)$. Then we have*

$$\begin{aligned}
&I^{\alpha,\eta}x(t)\Big(I^{\alpha,\eta}y(t)I^{\gamma,\zeta}(zfg)(t)+2I^{\alpha,\eta}z(t)I^{\gamma,\zeta}(yfg)(t)+I^{\gamma,\zeta}y(t)I^{\alpha,\eta}(zfg)(t)\Big)\\
&+\Big(I^{\alpha,\eta}y(t)I^{\gamma,\zeta}z(t)+I^{\gamma,\zeta}y(t)I^{\alpha,\eta}z(t)\Big)I^{\alpha,\eta}(xfg)(t)\geq I^{\alpha,\eta}x(t)\Big(I^{\alpha,\eta}(yf)(t)I^{\alpha,\eta}(zg)(t)+I^{\alpha,\eta}(zf)(t)I^{\gamma,\zeta}(yg)(t)\Big)\\
&+I^{\gamma,\zeta}y(t)\Big(I^{\alpha,\eta}(xf)(t)I^{\gamma,\zeta}(zg)(t)+I^{\gamma,\zeta}(zf)(t)I^{\alpha,\eta}(xg)(t)\Big)+I^{\alpha,\eta}z(t)\Big(I^{\alpha,\eta}(xf)(t)I^{\gamma,\zeta}(yg)(t)+I^{\gamma,\zeta}(yf)(t)I^{\alpha,\eta}(xg)(t)\Big).
\end{aligned}\tag{45}$$

for all $t>0$, $\alpha,\gamma>0$, $-1<\max(\eta,\zeta)<0$.

Nextly, we establish some new Grüss-type integral inequalities involving Saigo fractional integral operators.

Lemma 2.13. *Let f be an integrable function on $[0,\infty)$ satisfying the condition (7) on $[0,\infty)$ and let x be a continuous function on $[0,\infty)$. Then we have*

$$\begin{aligned}
I_{0,t}^{\alpha,\beta,\eta}x(t)I_{0,t}^{\alpha,\beta,\eta}(xf^2)(t)-\Big(I_{0,t}^{\alpha,\beta,\eta}(xf)(t)\Big)^2&=\Big(\Phi I_{0,t}^{\alpha,\beta,\eta}x(t)-I_{0,t}^{\alpha,\beta,\eta}(xf)(t)\Big)\Big(I_{0,t}^{\alpha,\beta,\eta}(xf)(t)-\phi I_{0,t}^{\alpha,\beta,\eta}x(t)\Big)\\
&\quad-I_{0,t}^{\alpha,\beta,\eta}x(t)I_{0,t}^{\alpha,\beta,\eta}\Big(x(t)(\Phi-f(t))(f(t)-\phi)\Big),
\end{aligned}\tag{46}$$

for all $t>0$, $\alpha>\max\{0,-\beta\}$, $\beta<1$, $\beta-1<\eta<0$.

Proof. Let f be an integrable function on $[0,\infty)$ satisfying the condition (7) on $[0,\infty)$. For any $\rho,\tau\in[0,\infty)$, we have

$$\begin{aligned}
&(\Phi-f(\rho))(f(\tau)-\phi)+(\Phi-f(\tau))(f(\rho)-\phi)-(\Phi-f(\tau))(f(\tau)-\phi)\\
&\qquad-(\Phi-f(\rho))(f(\rho)-\phi)=f^2(\tau)+f^2(\rho)-2f(\rho)f(\tau).
\end{aligned}\tag{47}$$

Multiplying both sides of (47) by $x(\rho)F(t,\rho)$ (defined by (21)), and integrating the resulting inequality obtained with respect to ρ from 0 to t, we have

$$\begin{aligned}
&(f(\tau)-\phi)\Big(\Phi I_{0,t}^{\alpha,\beta,\eta}x(t)-I_{0,t}^{\alpha,\beta,\eta}(xf)(t)\Big)+(\Phi-f(\tau))\Big(I_{0,t}^{\alpha,\beta,\eta}(xf)(t)-\phi I_{0,t}^{\alpha,\beta,\eta}x(t)\Big)-(\Phi-f(\tau))(f(\tau)-\phi)I_{0,t}^{\alpha,\beta,\eta}x(t)\\
&\qquad-I_{0,t}^{\alpha,\beta,\eta}\Big(x(t)(\Phi-f(t))(f(t)-\phi)\Big)=f^2(\tau)I_{0,t}^{\alpha,\beta,\eta}x(t)+I_{0,t}^{\alpha,\beta,\eta}(xf^2)(t)-2f(\tau)I_{0,t}^{\alpha,\beta,\eta}(xf)(t).
\end{aligned}\tag{48}$$

Multiplying both sides of (48) by $x(\tau)F(t,\tau)$, and integrating the resulting inequality obtained with respect to τ from 0 to t, we have

$$\begin{aligned}
&\Big(I_{0,t}^{\alpha,\beta,\eta}(xf)(t)-\phi I_{0,t}^{\alpha,\beta,\eta}x(t)\Big)\Big(\Phi I_{0,t}^{\alpha,\beta,\eta}x(t)-I_{0,t}^{\alpha,\beta,\eta}(xf)(t)\Big)+\Big(\Phi I_{0,t}^{\alpha,\beta,\eta}x(t)-I_{0,t}^{\alpha,\beta,\eta}(xf)(t)\Big)\Big(I_{0,t}^{\alpha,\beta,\eta}(xf)(t)-\phi I_{0,t}^{\alpha,\beta,\eta}x(t)\Big)\\
&-I_{0,t}^{\alpha,\beta,\eta}\Big(x(t)(\Phi-f(t))(f(t)-\phi)\Big)I_{0,t}^{\alpha,\beta,\eta}x(t)-I_{0,t}^{\alpha,\beta,\eta}x(t)I_{0,t}^{\alpha,\beta,\eta}\Big(x(t)(\Phi-f(t))(f(t)-\phi)\Big)\\
&\qquad=I_{0,t}^{\alpha,\beta,\eta}(xf^2)(t)I_{0,t}^{\alpha,\beta,\eta}x(t)+I_{0,t}^{\alpha,\beta,\eta}x(t)I_{0,t}^{\alpha,\beta,\eta}(xf^2)(t)-2I_{0,t}^{\alpha,\beta,\eta}(xf)(t)I_{0,t}^{\alpha,\beta,\eta}(xf)(t),
\end{aligned}\tag{49}$$

which gives (46) and proves the lemma. □

Theorem 2.14. *Let f and g be two integrable functions satisfying the condition (7) on $[0,\infty)$ and let x be a nonnegative continuous function on $[0,\infty)$. Then we have*

$$\left|I_{0,t}^{\alpha,\beta,\eta}x(t)I_{0,t}^{\alpha,\beta,\eta}(xfg)(t)-I_{0,t}^{\alpha,\beta,\eta}(xf)(t)I_{0,t}^{\alpha,\beta,\eta}(xg)(t)\right|\leq\frac{1}{4}(\Phi-\phi)(\Psi-\psi)\left(I_{0,t}^{\alpha,\beta,\eta}x(t)\right)^2, \tag{50}$$

for all $t>0$, $\alpha>\max\{0,-\beta\}$, $\beta<1$, $\beta-1<\eta<0$.

Proof. Let f and g be two functions satisfying the conditions of Theorem 2.14. Let $H(\tau,\rho)$ be defined by

$$H(\tau,\rho)=(f(\tau)-f(\rho))(g(\tau)-g(\rho)),\quad \tau,\rho\in[0,t],\quad t>0. \tag{51}$$

Multiplying both sides of (51) by $x(\tau)F(t,\tau)x(\rho)F(t,\rho)$ and integrating the resulting identity with respect to τ and ρ from 0 to t, we can state that

$$\int_0^t\int_0^t x(\tau)F(t,\tau)x(\rho)F(t,\rho)H(\tau,\rho)d\tau d\rho=2I_{0,t}^{\alpha,\beta,\eta}x(t)I_{0,t}^{\alpha,\beta,\eta}(xfg)(t)-2I_{0,t}^{\alpha,\beta,\eta}(xf)(t)I_{0,t}^{\alpha,\beta,\eta}(xg)(t). \tag{52}$$

Thanks to the weighted Cauchy-Schwartz integral inequality for double integrals, we can write that

$$\begin{aligned}&\left(\int_0^t\int_0^t x(\tau)F(t,\tau)x(\rho)F(t,\rho)H(\tau,\rho)d\tau d\rho\right)^2\\&\leq\left(\int_0^t\int_0^t x(\tau)F(t,\tau)x(\rho)F(t,\rho)(f(\tau)-f(\rho))d\tau d\rho\right)\left(\int_0^t\int_0^t x(\tau)F(t,\tau)x(\rho)F(t,\rho)(g(\tau)-g(\rho))d\tau d\rho\right)\\&=4\left(I_{0,t}^{\alpha,\beta,\eta}x(t)I_{0,t}^{\alpha,\beta,\eta}(xf^2)(t)-\left(I_{0,t}^{\alpha,\beta,\eta}(xf)(t)\right)^2\right)\left(I_{0,t}^{\alpha,\beta,\eta}x(t)I_{0,t}^{\alpha,\beta,\eta}(xg^2)(t)-\left(I_{0,t}^{\alpha,\beta,\eta}(xg)(t)\right)^2\right).\end{aligned} \tag{53}$$

Since $(\Phi-f(\tau))(f(\tau)-\phi)\geq 0$ and $(\Psi-g(\tau))(g(\tau)-\psi)\geq 0$, we have

$$I_{0,t}^{\alpha,\beta,\eta}x(t)I_{0,t}^{\alpha,\beta,\eta}\left(x(t)(\Phi-f(t))(f(t)-\phi)\right)\geq 0, \tag{54}$$

and

$$I_{0,t}^{\alpha,\beta,\eta}x(t)I_{0,t}^{\alpha,\beta,\eta}\left(x(t)(\Psi-g(t))(g(t)-\psi)\right)\geq 0. \tag{55}$$

Thus, from (54), (55) and Lemma 2.13,we get

$$I_{0,t}^{\alpha,\beta,\eta}x(t)I_{0,t}^{\alpha,\beta,\eta}(xf^2)(t)-\left(I_{0,t}^{\alpha,\beta,\eta}(xf)(t)\right)^2\leq\left(\Phi I_{0,t}^{\alpha,\beta,\eta}x(t)-I_{0,t}^{\alpha,\beta,\eta}(xf)(t)\right)\left(I_{0,t}^{\alpha,\beta,\eta}(xf)(t)-\phi I_{0,t}^{\alpha,\beta,\eta}x(t)\right), \tag{56}$$

and

$$I_{0,t}^{\alpha,\beta,\eta}x(t)I_{0,t}^{\alpha,\beta,\eta}(xg^2)(t)-\left(I_{0,t}^{\alpha,\beta,\eta}(xg)(t)\right)^2\leq\left(\Psi I_{0,t}^{\alpha,\beta,\eta}x(t)-I_{0,t}^{\alpha,\beta,\eta}(xg)(t)\right)\left(I_{0,t}^{\alpha,\beta,\eta}(xg)(t)-\psi I_{0,t}^{\alpha,\beta,\eta}x(t)\right). \tag{57}$$

Combining (52), (53), (56) and (57), we deduce that

$$\begin{aligned}&\left(I_{0,t}^{\alpha,\beta,\eta}x(t)I_{0,t}^{\alpha,\beta,\eta}(xfg)(t)-I_{0,t}^{\alpha,\beta,\eta}(xf)(t)I_{0,t}^{\alpha,\beta,\eta}(xg)(t)\right)^2\leq\left(\Phi I_{0,t}^{\alpha,\beta,\eta}x(t)-I_{0,t}^{\alpha,\beta,\eta}(xf)(t)\right)\\&\times\left(I_{0,t}^{\alpha,\beta,\eta}(xf)(t)-\phi I_{0,t}^{\alpha,\beta,\eta}x(t)\right)\left(\Psi I_{0,t}^{\alpha,\beta,\eta}x(t)-I_{0,t}^{\alpha,\beta,\eta}(xg)(t)\right)\left(I_{0,t}^{\alpha,\beta,\eta}(xg)(t)-\psi I_{0,t}^{\alpha,\beta,\eta}x(t)\right).\end{aligned} \tag{58}$$

Now using the elementary inequality $4xy \le (x+y)^2$, $x, y \in \mathbb{R}$, we can state that

$$4\Big(\Phi I_{0,t}^{\alpha,\beta,\eta}x(t) - I_{0,t}^{\alpha,\beta,\eta}(xf)(t)\Big)\Big(I_{0,t}^{\alpha,\beta,\eta}(xf)(t) - \phi I_{0,t}^{\alpha,\beta,\eta}x(t)\Big) \le \Big(I_{0,t}^{\alpha,\beta,\eta}x(t)(\Phi-\phi)\Big)^2, \tag{59}$$

and

$$4\Big(\Psi I_{0,t}^{\alpha,\beta,\eta}x(t) - I_{0,t}^{\alpha,\beta,\eta}(xg)(t)\Big)\Big(I_{0,t}^{\alpha,\beta,\eta}(xg)(t) - \psi I_{0,t}^{\alpha,\beta,\eta}x(t)\Big) \le \Big(I_{0,t}^{\alpha,\beta,\eta}x(t)(\Psi-\psi)\Big)^2. \tag{60}$$

From (58)-(60), we abtain (50). This complete the proof of Theorem 2.14. □

Lemma 2.15. *Let f and g be two integrable functions on $[0,\infty)$ and let x and y be two nonnegative continuous functions on $[0,\infty)$. Then we have*

$$\begin{aligned}
&\Big(I_{0,t}^{\alpha,\beta,\eta}x(t)I_{0,t}^{\gamma,\delta,\zeta}(yfg)(t) + I_{0,t}^{\gamma,\delta,\zeta}y(t)I_{0,t}^{\alpha,\beta,\eta}(xfg)(t) - I_{0,t}^{\alpha,\beta,\eta}(xf)(t)I_{0,t}^{\gamma,\delta,\zeta}(yg)(t) - I_{0,t}^{\gamma,\delta,\zeta}(yf)(t)I_{0,t}^{\alpha,\beta,\eta}(xg)(t)\Big)^2\\
&\le \Big(I_{0,t}^{\alpha,\beta,\eta}x(t)I_{0,t}^{\gamma,\delta,\zeta}(yf^2)(t) + I_{0,t}^{\gamma,\delta,\zeta}y(t)I_{0,t}^{\alpha,\beta,\eta}(xf^2)(t) - 2I_{0,t}^{\alpha,\beta,\eta}(xf)(t)I_{0,t}^{\gamma,\delta,\zeta}(yf)(t)\Big)\\
&\qquad\times\Big(I_{0,t}^{\alpha,\beta,\eta}x(t)I_{0,t}^{\gamma,\delta,\zeta}(yg^2)(t) + I_{0,t}^{\gamma,\delta,\zeta}y(t)I_{0,t}^{\alpha,\beta,\eta}(xg^2)(t) - 2I_{0,t}^{\alpha,\beta,\eta}(xg)(t)I_{0,t}^{\gamma,\delta,\zeta}(yg)(t)\Big),
\end{aligned} \tag{61}$$

for all $t>0$, $\alpha > \max\{0,-\beta\}$, $\gamma > \max\{0,-\delta\}$, $\beta,\delta<1$, $\beta-1<\eta<0$, $\delta-1<\zeta<0$.

Proof. Multiplying (51) by $x(\tau)F(t,\tau)y(\rho)G(t,\rho)$ ($F(t,\tau)$ and $G(t,\rho)$ defined by (21) and (33), respectively) and integrating the resulting identity with respect to τ and ρ from 0 to t, we can get

$$\begin{aligned}
\int_0^t\int_0^t x(\tau)F(t,\tau)y(\rho)G(t,\rho)H(\tau,\rho)d\tau d\rho &= I_{0,t}^{\alpha,\beta,\eta}x(t)I_{0,t}^{\gamma,\delta,\zeta}(yfg)(t) + I_{0,t}^{\gamma,\delta,\zeta}y(t)I_{0,t}^{\alpha,\beta,\eta}(xfg)(t)\\
&\quad - I_{0,t}^{\alpha,\beta,\eta}(xf)(t)I_{0,t}^{\gamma,\delta,\zeta}(yg)(t) - I_{0,t}^{\gamma,\delta,\zeta}(yf)(t)I_{0,t}^{\alpha,\beta,\eta}(xg)(t).
\end{aligned} \tag{62}$$

Then, thanks to the weighted Cauchy-Schwartz integral inequality for double integrals, we can obtain (61). □

Lemma 2.16. *Let f be an integrable function on $[0,\infty)$ and let x and y be two nonnegative continuous functions on $[0,\infty)$. Then we have*

$$\begin{aligned}
&I_{0,t}^{\alpha,\beta,\eta}x(t)I_{0,t}^{\gamma,\delta,\zeta}(yf^2)(t) + I_{0,t}^{\gamma,\delta,\zeta}y(t)I_{0,t}^{\alpha,\beta,\eta}(xf^2)(t) - 2I_{0,t}^{\alpha,\beta,\eta}(xf)(t)I_{0,t}^{\gamma,\delta,\zeta}(yf)(t)\\
&= \Big(\Phi I_{0,t}^{\alpha,\beta,\eta}x(t) - I_{0,t}^{\alpha,\beta,\eta}(xf)(t)\Big)\Big(I_{0,t}^{\gamma,\delta,\zeta}(yf)(t) - \phi I_{0,t}^{\gamma,\delta,\zeta}y(t)\Big) + \Big(I_{0,t}^{\alpha,\beta,\eta}(xf)(t) - \phi I_{0,t}^{\alpha,\beta,\eta}x(t)\Big)\Big(\Phi I_{0,t}^{\gamma,\delta,\zeta}y(t) - I_{0,t}^{\gamma,\delta,\zeta}(yf)(t)\Big)\\
&\quad - I_{0,t}^{\alpha,\beta,\eta}x(t)I_{0,t}^{\gamma,\delta,\zeta}\Big(y(t)(\Phi-f(t))(f(t)-\phi)\Big) - I_{0,t}^{\gamma,\delta,\zeta}y(t)I_{0,t}^{\alpha,\beta,\eta}\Big(x(t)(\Phi-f(t))(f(t)-\phi)\Big),
\end{aligned} \tag{63}$$

for all $t>0$, $\alpha > \max\{0,-\beta\}$, $\gamma > \max\{0,-\delta\}$, $\beta,\delta<1$, $\beta-1<\eta<0$, $\delta-1<\zeta<0$.

Proof. Multiplying both sides of (48) by $y(\tau)G(t,\tau)$, and integrating the resulting inequality obtained with respect to τ from 0 to t, we have

$$\begin{aligned}
&\Big(I_{0,t}^{\gamma,\delta,\zeta}(yf)(t) - \phi I_{0,t}^{\gamma,\delta,\zeta}y(t)\Big)\Big(\Phi I_{0,t}^{\alpha,\beta,\eta}x(t) - I_{0,t}^{\alpha,\beta,\eta}(xf)(t)\Big) + \Big(\Phi I_{0,t}^{\gamma,\delta,\zeta}y(t) - I_{0,t}^{\gamma,\delta,\zeta}(yf)(t)\Big)\Big(I_{0,t}^{\alpha,\beta,\eta}(xf)(t) - \phi I_{0,t}^{\alpha,\beta,\eta}x(t)\Big)\\
&- I_{0,t}^{\gamma,\delta,\zeta}\Big(y(t)(\Phi-f(t))(f(t)-\phi)\Big)I_{0,t}^{\alpha,\beta,\eta}x(t) - I_{0,t}^{\gamma,\delta,\zeta}y(t)I_{0,t}^{\alpha,\beta,\eta}\Big(x(t)(\Phi-f(t))(f(t)-\phi)\Big)\\
&\qquad = I_{0,t}^{\gamma,\delta,\zeta}(yf^2)(t)I_{0,t}^{\alpha,\beta,\eta}x(t) + I_{0,t}^{\gamma,\delta,\zeta}y(t)I_{0,t}^{\alpha,\beta,\eta}(xf^2)(t) - 2I_{0,t}^{\gamma,\delta,\zeta}(yf)(t)I_{0,t}^{\alpha,\beta,\eta}(xf)(t),
\end{aligned} \tag{64}$$

which gives (63) and proves the lemma. □

Theorem 2.17. *Let f and g be two integrable functions satisfying the condition (7) on $[0,\infty)$ and let x and y be two nonnegative continuous functions on $[0,\infty)$. Then we have*

$$\begin{aligned}&\left(I_{0,t}^{\alpha,\beta,\eta}x(t)I_{0,t}^{\gamma,\delta,\zeta}(yfg)(t)+I_{0,t}^{\gamma,\delta,\zeta}y(t)I_{0,t}^{\alpha,\beta,\eta}(xfg)(t)-I_{0,t}^{\alpha,\beta,\eta}(xf)(t)I_{0,t}^{\gamma,\delta,\zeta}(yg)(t)-I_{0,t}^{\gamma,\delta,\zeta}(yf)(t)I_{0,t}^{\alpha,\beta,\eta}(xg)(t)\right)^2\\ &\le\left[\left(\Phi I_{0,t}^{\alpha,\beta,\eta}x(t)-I_{0,t}^{\alpha,\beta,\eta}(xf)(t)\right)\left(I_{0,t}^{\gamma,\delta,\zeta}(yf)(t)-\phi I_{0,t}^{\gamma,\delta,\zeta}y(t)\right)+\left(I_{0,t}^{\alpha,\beta,\eta}(xf)(t)-\phi I_{0,t}^{\alpha,\beta,\eta}x(t)\right)\times\left(\Phi I_{0,t}^{\gamma,\delta,\zeta}y(t)-I_{0,t}^{\gamma,\delta,\zeta}(yf)(t)\right)\right]\\ &\times\left[\left(\Psi I_{0,t}^{\alpha,\beta,\eta}x(t)-I_{0,t}^{\alpha,\beta,\eta}(xg)(t)\right)\left(I_{0,t}^{\gamma,\delta,\zeta}(yg)(t)-\psi I_{0,t}^{\gamma,\delta,\zeta}y(t)\right)+\left(I_{0,t}^{\alpha,\beta,\eta}(xg)(t)-\psi I_{0,t}^{\alpha,\beta,\eta}x(t)\right)\left(\Psi I_{0,t}^{\gamma,\delta,\zeta}y(t)-I_{0,t}^{\gamma,\delta,\zeta}(yg)(t)\right)\right],\end{aligned}\tag{65}$$

for all $t>0$, $\alpha>\max\{0,-\beta\}$, $\gamma>\max\{0,-\delta\}$, $\beta,\delta<1$, $\beta-1<\eta<0$, $\delta-1<\zeta<0$.

Proof. Since $(\Phi-f(\tau))(f(\tau)-\phi)\ge 0$ and $(\Psi-g(\tau))(g(\tau)-\psi)\ge 0$, we have

$$-I_{0,t}^{\alpha,\beta,\eta}x(t)I_{0,t}^{\gamma,\delta,\zeta}\left(y(t)(\Phi-f(t))(f(t)-\phi)\right)-I_{0,t}^{\gamma,\delta,\zeta}y(t)I_{0,t}^{\alpha,\beta,\eta}\left(x(t)(\Phi-f(t))(f(t)-\phi)\right)\le 0,\tag{66}$$

and

$$-I_{0,t}^{\alpha,\beta,\eta}x(t)I_{0,t}^{\gamma,\delta,\zeta}\left(y(t)(\Psi-g(t))(g(t)-\psi)\right)-I_{0,t}^{\gamma,\delta,\zeta}y(t)I_{0,t}^{\alpha,\beta,\eta}\left(x(t)(\Psi-g(t))(g(t)-\psi)\right)\le 0,\tag{67}$$

Applying Lemma 2.16 to f and g, and using Lemma 2.15 and the formulas (66), (67), we obtain (65). □

Theorem 2.18. *Let f and g be two integrable functions satisfying the condition (7) on $[0,\infty)$ and let x and y be two nonnegative continuous functions on $[0,\infty)$. Then we have*

$$\begin{aligned}&\left|I_{0,t}^{\alpha,\beta,\eta}x(t)I_{0,t}^{\gamma,\delta,\zeta}(yfg)(t)+I_{0,t}^{\gamma,\delta,\zeta}y(t)I_{0,t}^{\alpha,\beta,\eta}(xfg)(t)-I_{0,t}^{\alpha,\beta,\eta}(xf)(t)I_{0,t}^{\gamma,\delta,\zeta}(yg)(t)-I_{0,t}^{\gamma,\delta,\zeta}(yf)(t)I_{0,t}^{\alpha,\beta,\eta}(xg)(t)\right|\\ &\qquad\le I_{0,t}^{\alpha,\beta,\eta}x(t)I_{0,t}^{\gamma,\delta,\zeta}y(t)(\Phi-\phi)(\Psi-\psi),\end{aligned}\tag{68}$$

for all $t>0$, $\alpha>\max\{0,-\beta\}$, $\gamma>\max\{0,-\delta\}$, $\beta,\delta<1$, $\beta-1<\eta<0$, $\delta-1<\zeta<0$.

Proof. From the condition (7), we have

$$|f(\tau)-f(\rho)|\le\Phi-\phi,\quad |g(\tau)-g(\rho)|\le\Psi-\psi,\quad \tau,\rho\in[0,\infty),\tag{69}$$

which implies that

$$|H(\tau,\rho)|=|f(\tau)-f(\rho)||g(\tau)-g(\rho)|\le(\Phi-\phi)(\Psi-\psi).\tag{70}$$

Combining (62) and (70), we obtain that

$$\begin{aligned}&\left|I_{0,t}^{\alpha,\beta,\eta}x(t)I_{0,t}^{\gamma,\delta,\zeta}(yfg)(t)+I_{0,t}^{\gamma,\delta,\zeta}y(t)I_{0,t}^{\alpha,\beta,\eta}(xfg)(t)-I_{0,t}^{\alpha,\beta,\eta}(xf)(t)I_{0,t}^{\gamma,\delta,\zeta}(yg)(t)-I_{0,t}^{\gamma,\delta,\zeta}(yf)(t)I_{0,t}^{\alpha,\beta,\eta}(xg)(t)\right|\\ &\le\int_0^t\int_0^t x(\tau)F(t,\tau)y(\rho)G(t,\rho)|H(\tau,\rho)|d\tau d\rho\\ &\qquad\le\int_0^t\int_0^t x(\tau)F(t,\tau)y(\rho)G(t,\rho)d\tau d\rho(\Phi-\phi)(\Psi-\psi)=I_{0,t}^{\alpha,\beta,\eta}x(t)I_{0,t}^{\gamma,\delta,\zeta}y(t)(\Phi-\phi)(\Psi-\psi).\end{aligned}\tag{71}$$

This ends the proof. □

Theorem 2.19. *Let f and g be two integrable functions satisfying the condition (12) on $[0,\infty)$ and let x and y be two nonnegative continuous functions on $[0,\infty)$. Then we have*

$$\left|I_{0,t}^{\alpha,\beta,\eta}x(t)I_{0,t}^{\gamma,\delta,\zeta}(yfg)(t)+I_{0,t}^{\gamma,\delta,\zeta}y(t)I_{0,t}^{\alpha,\beta,\eta}(xfg)(t)-I_{0,t}^{\alpha,\beta,\eta}(xf)(t)I_{0,t}^{\gamma,\delta,\zeta}(yg)(t)-I_{0,t}^{\gamma,\delta,\zeta}(yf)(t)I_{0,t}^{\alpha,\beta,\eta}(xg)(t)\right| \leq M\Big(I_{0,t}^{\alpha,\beta,\eta}x(t)I_{0,t}^{\gamma,\delta,\zeta}(yg^2)(t)+I_{0,t}^{\gamma,\delta,\zeta}y(t)I_{0,t}^{\alpha,\beta,\eta}(xg^2)(t)-2I_{0,t}^{\alpha,\beta,\eta}(xg)(t)I_{0,t}^{\gamma,\delta,\zeta}(yg)(t)\Big), \quad (72)$$

for all $t>0$, $\alpha>\max\{0,-\beta\}$, $\gamma>\max\{0,-\delta\}$, $\beta,\delta<1$, $\beta-1<\eta<0$, $\delta-1<\zeta<0$.

Proof. From the condition (12), we have

$$|f(\tau)-f(\rho)|\leq M|g(\tau)-g(\rho)|, \quad \tau,\rho\in[0,\infty), \quad (73)$$

which implies that

$$|H(\tau,\rho)|=|f(\tau)-f(\rho)||g(\tau)-g(\rho)|\leq M(g(\tau)-g(\rho))^2. \quad (74)$$

Combining (62) and (74), we get that

$$\begin{aligned}&\left|I_{0,t}^{\alpha,\beta,\eta}x(t)I_{0,t}^{\gamma,\delta,\zeta}(yfg)(t)+I_{0,t}^{\gamma,\delta,\zeta}y(t)I_{0,t}^{\alpha,\beta,\eta}(xfg)(t)-I_{0,t}^{\alpha,\beta,\eta}(xf)(t)I_{0,t}^{\gamma,\delta,\zeta}(yg)(t)-I_{0,t}^{\gamma,\delta,\zeta}(yf)(t)I_{0,t}^{\alpha,\beta,\eta}(xg)(t)\right|\\ &\leq\int_0^t\int_0^t x(\tau)F(t,\tau)y(\rho)G(t,\rho)|H(\tau,\rho)|d\tau d\rho\leq M\int_0^t\int_0^t x(\tau)F(t,\tau)y(\rho)G(t,\rho)(g(\tau)-g(\rho))^2d\tau d\rho\\ &=M\Big(I_{0,t}^{\alpha,\beta,\eta}x(t)I_{0,t}^{\gamma,\delta,\zeta}(yg^2)(t)+I_{0,t}^{\gamma,\delta,\zeta}y(t)I_{0,t}^{\alpha,\beta,\eta}(xg^2)(t)-2I_{0,t}^{\alpha,\beta,\eta}(xg)(t)I_{0,t}^{\gamma,\delta,\zeta}(yg)(t)\Big). \quad (75)\end{aligned}$$

This ends the proof. □

Theorem 2.20. *Let f and g be two integrable functions on $[0,\infty)$ satisfying the lipschitzian condition with the constants L_1 and L_2 and let x and y be two nonnegative continuous functions on $[0,\infty)$. Then we have*

$$\left|I_{0,t}^{\alpha,\beta,\eta}x(t)I_{0,t}^{\gamma,\delta,\zeta}(yfg)(t)+I_{0,t}^{\gamma,\delta,\zeta}y(t)I_{0,t}^{\alpha,\beta,\eta}(xfg)(t)-I_{0,t}^{\alpha,\beta,\eta}(xf)(t)I_{0,t}^{\gamma,\delta,\zeta}(yg)(t)-I_{0,t}^{\gamma,\delta,\zeta}(yf)(t)I_{0,t}^{\alpha,\beta,\eta}(xg)(t)\right| \leq L_1L_2\Big(I_{0,t}^{\alpha,\beta,\eta}x(t)I_{0,t}^{\gamma,\delta,\zeta}(t^2y(t))+I_{0,t}^{\gamma,\delta,\zeta}y(t)I_{0,t}^{\alpha,\beta,\eta}(t^2x(t))-2I_{0,t}^{\alpha,\beta,\eta}(tx(t))I_{0,t}^{\gamma,\delta,\zeta}(ty(t))\Big), \quad (76)$$

for all $t>0$, $\alpha>\max\{0,-\beta\}$, $\gamma>\max\{0,-\delta\}$, $\beta,\delta<1$, $\beta-1<\eta<0$, $\delta-1<\zeta<0$.

Proof. From the conditions of Theorem 2.20, we have

$$|f(\tau)-f(\rho)|\leq L_1|\tau-\rho|, \quad |g(\tau)-g(\rho)|\leq L_2|\tau-\rho|, \quad \tau,\rho\in[0,\infty), \quad (77)$$

which implies that

$$|H(\tau,\rho)|=|f(\tau)-f(\rho)||g(\tau)-g(\rho)|\leq L_1L_2(\tau-\rho)^2. \quad (78)$$

Combining (62) and (78), we get that

$$\begin{aligned}&\left|I_{0,t}^{\alpha,\beta,\eta}x(t)I_{0,t}^{\gamma,\delta,\zeta}(yfg)(t)+I_{0,t}^{\gamma,\delta,\zeta}y(t)I_{0,t}^{\alpha,\beta,\eta}(xfg)(t)-I_{0,t}^{\alpha,\beta,\eta}(xf)(t)I_{0,t}^{\gamma,\delta,\zeta}(yg)(t)-I_{0,t}^{\gamma,\delta,\zeta}(yf)(t)I_{0,t}^{\alpha,\beta,\eta}(xg)(t)\right|\\ &\leq\int_0^t\int_0^t x(\tau)F(t,\tau)y(\rho)G(t,\rho)|H(\tau,\rho)|d\tau d\rho\leq L_1L_2\int_0^t\int_0^t x(\tau)F(t,\tau)y(\rho)G(t,\rho)(\tau-\rho)^2d\tau d\rho\\ &=L_1L_2\Big(I_{0,t}^{\alpha,\beta,\eta}x(t)I_{0,t}^{\gamma,\delta,\zeta}(t^2y(t))+I_{0,t}^{\gamma,\delta,\zeta}y(t)I_{0,t}^{\alpha,\beta,\eta}(t^2x(t))-2I_{0,t}^{\alpha,\beta,\eta}(tx(t))I_{0,t}^{\gamma,\delta,\zeta}(ty(t))\Big). \quad (79)\end{aligned}$$

This ends the proof. □

Corollary 2.21. *Let f and g be two differentiable functions on $[0,\infty)$ and let x and y be two nonnegative continuous functions on $[0,\infty)$. Then we have*

$$\left|I_{0,t}^{\alpha,\beta,\eta}x(t)I_{0,t}^{\gamma,\delta,\zeta}(yfg)(t)+I_{0,t}^{\gamma,\delta,\zeta}y(t)I_{0,t}^{\alpha,\beta,\eta}(xfg)(t)-I_{0,t}^{\alpha,\beta,\eta}(xf)(t)I_{0,t}^{\gamma,\delta,\zeta}(yg)(t)-I_{0,t}^{\gamma,\delta,\zeta}(yf)(t)I_{0,t}^{\alpha,\beta,\eta}(xg)(t)\right|$$
$$\leq \|f'\|_\infty\|g'\|_\infty\left(I_{0,t}^{\alpha,\beta,\eta}x(t)I_{0,t}^{\gamma,\delta,\zeta}(t^2y(t))+I_{0,t}^{\gamma,\delta,\zeta}y(t)I_{0,t}^{\alpha,\beta,\eta}(t^2x(t))-2I_{0,t}^{\alpha,\beta,\eta}(tx(t))I_{0,t}^{\gamma,\delta,\zeta}(ty(t))\right), \quad (80)$$

for all $t>0$, $\alpha>\max\{0,-\beta\}$, $\gamma>\max\{0,-\delta\}$, $\beta,\delta<1$, $\beta-1<\eta<0$, $\delta-1<\zeta<0$.

Proof. We have $f(\tau)-f(\rho)=\int_\rho^\tau f'(t)dt$ and $g(\tau)-g(\rho)=\int_\rho^\tau g'(t)dt$. That is, $|f(\tau)-f(\rho)|\leq\|f'\|_\infty|\tau-\rho|$, $|g(\tau)-g(\rho)|\leq\|g'\|_\infty|\tau-\rho|$, $\tau,\rho\in[0,\infty)$, and the result follows from Theorem 2.20. This ends the proof. □

Remark 2.22. If we replace β by $-\alpha$ and δ by $-\gamma$, set $x(t)=y(t)=1$, and make use of the relation (17), then Theorems 2.14 and 2.17 correspond to the known results due to Dahmani et al. [16]. If we replace β by $-\alpha$ and δ by $-\gamma$, and make use of the relation (17), then Theorems 2.18 and 2.20 and Theorem 2.19 and Ccorollary 2.21 immediately reduce to the known results due to Dahmani et al. [16] and Dahmani [14], respectively.

Remark 2.23. Similar to Ccorollary 2.9–2.12, by putting $\beta=0$ and $\delta=0$, and using the relation (18), Lemmas 2.13, 2.15, 2.16 and Theorems 2.14, 2.17–2.20 can also yield some new fractional integral inequalities involving the Erdélyi-Kober type fractional integral operators defined by (18).

3. Saigo Fractional q-integral Inequalities

For the convenience of the reader, we firstly deem it proper to give here basic definitions and related details of the fractional q-calculus, we can see [25].

For any complex number $\alpha\in\mathbb{C}$, we define (Notation in q-Calculus [20, 27])

$$[\alpha]_q=\frac{1-q^\alpha}{1-q},\quad q\neq1;\quad [n]_q!=[n]_q[n-1]_q\cdots[2]_q[1]_q,\quad n\in\mathbb{N} \quad (81)$$

and

$$([\lambda]_q)_n=[\lambda]_q[\lambda+1]_q\cdots[\lambda+n-1]_q,\quad (n\in\mathbb{N},\ \lambda\in\mathbb{C}), \quad (82)$$

with $[0]_q!=1$ and the q-shifted factorial is defined for as a product of n factors by

$$(\alpha;q)_n=1,\ n=0;\quad (\alpha;q)_n=(1-\alpha)(1-\alpha q)\cdots(1-\alpha q^{n-1}),\ n\in\mathbb{N}, \quad (83)$$

and in terms of the basic analogue of the gamma function

$$(q^\alpha;q)_n=\frac{\Gamma_q(\alpha+n)(1-q)^n}{\Gamma_q(\alpha)}\quad (n>0), \quad (84)$$

where the q-gamma function is defined by

$$\Gamma_q(t)=\frac{(q;q)_\infty(1-q)^{1-t}}{(q^t;q)_\infty}\quad (0<q<1). \quad (85)$$

We note that

$$\Gamma_q(1+t)=\frac{(1-q)^t\Gamma_q(t)}{1-q}, \quad (86)$$

and if $|q| < 1$, the definition (83) remains meaningful for $n = \infty$, as a convergent infinite product given by

$$(\alpha; q)_\infty = \prod_{j=0}^{\infty}(1 - \alpha q^j), \tag{87}$$

For $\lambda, \mu \in \mathbb{C}$, we have (see [29, p.435-436])

$$(\lambda; q)_\mu = \prod_{j=0}^{\infty} \frac{1 - \lambda q^j}{1 - \lambda q^{\mu+j}}, \quad (q \in \mathbb{C},\ |q| < 1), \tag{88}$$

$$\lim_{q\to 1} \frac{(q^\lambda; q)_n}{(q^\mu; q)_n} = \frac{(\lambda)_n}{(\mu)_n}, \quad (n \in \mathbb{N}_0, \mu \notin \mathbb{N}_0 := \{0, -1, -2, \ldots\}), \tag{89}$$

and

$$\lim_{q\to 1}[\alpha]_q = \alpha, \quad \lim_{q\to 1}[n]_q! = n!, \quad \lim_{q\to 1}([\lambda]_q)_n = (\lambda)_n, \tag{90}$$

where $(\lambda)_\nu$ denotes the Pochammer symbol (or the shifted or rising factorial) defined, in terms of the familiar Gamma function, by

$$(\lambda)_\nu = \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} = \begin{cases} 1, & \nu = 0;\ \ \lambda \in \mathbb{C} \setminus \{0\}, \\ \lambda(\lambda + 1)\ldots(\lambda + n - 1), & \nu = n \in \mathbb{N};\ \ \lambda \in \mathbb{C}, \end{cases} \tag{91}$$

it being understood conventionally that $(0)_0 := 1$.

The q-binomial expansion is also given by

$$(x - y)_\mu = x^\mu(-y/x; q)_\mu = x^\mu \prod_{j=0}^{\infty}\left(\frac{1 - (y/x)q^j}{1 - (y/x)q^{\mu+j}}\right). \tag{92}$$

Let $t_0 \in \mathbb{R}$, then we define a specific time scale

$$\mathbb{T}_{t_0} = \{t; t = t_0 q^n,\ n \text{ a nonnegative integer}\} \cup \{0\}, \quad 0 < q < 1, \tag{93}$$

and for convenience sake, we denote $\mathbb{T}_{t_0}$ by $\mathbb{T}$ throughout this paper.

The Jackson's q-derivative and q-integral of a function f defined on $\mathbb{T}$ are, respectively, given by

$$D_q f(t) = \frac{f(t) - f(qt)}{t(1 - q)} \qquad (t \neq 0, \quad q \neq 1) \tag{94}$$

and

$$\int_0^t f(\tau) d_q\tau = t(1 - q)\sum_{k=0}^{\infty} q^k f(tq^k). \tag{95}$$

Definition 3.1 ([1]). The Riemann-Liouville fractional q-integral operator of a function $f(t)$ of order α is given by

$$I_q^\alpha f(t) = \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^t (q\tau/t; q)_{\alpha-1} f(\tau) d_q\tau \qquad (\alpha > 0, \quad 0 < q < 1) \tag{96}$$

where

$$(a; q)_\alpha = \frac{(a; q)_\infty}{(aq^\alpha; q)_\infty} \qquad (\alpha \in \mathbb{R}). \tag{97}$$

Definition 3.2 ([2]). For $\alpha > 0$, $\eta \in \mathbb{R}$ and $0 < q < 1$, the basic analogue of the Kober fractional integral operator is given by

$$I_q^{\alpha,\eta} f(t) = \frac{t^{-\eta-1}}{\Gamma_q(\alpha)} \int_0^t (q\tau/t; q)_{\alpha-1} f(\tau) d_q\tau. \tag{98}$$

Definition 3.3 ([26]). For $\alpha > 0$, $\beta \in \mathbb{R}$, a basic analogue of the Saigo's fractional integral operator is given for $|\tau/t| < 1$ by

$$I_q^{\alpha,\beta,\eta} f(t) = \frac{t^{-\beta-1} q^{-\eta(\alpha+\beta)}}{\Gamma_q(\alpha)} \times \int_0^t (q\tau/t; q)_{\alpha-1} \mathscr{T}_{q,\frac{q^{\alpha+1}\tau}{t}} \Big({}_2\Phi_1[q^{\alpha+\beta}, q^{-\eta}; q^\alpha; q, q] \Big) f(\tau) d_q\tau. \tag{99}$$

where η is any non-negative integer, and the function ${}_2\Phi_1(-)$ and the q-translation operator occurring in the right-hand side of (99) are, respectively, defined by

$${}_2\Phi_1[a, b; c; q, t] = \sum_{n=0}^{\infty} \frac{(a; q)_n (b; q)_n}{(c; q)_n (q; q)_n} t^n \qquad (|q| < 1, \quad |t| < 1) \tag{100}$$

and

$$\mathscr{T}_{q,\tau}(f(t)) = \sum_{n=-\infty}^{+\infty} A_n t^n (\tau/t; q)_n, \tag{101}$$

where $(A_n)_{n\in\mathbb{Z}}$ $(\mathbb{Z} = 0, \pm1, \pm2, \cdots)$ is any bounded sequence of real or complex numbers.

For $f(t) = t^\mu$ in (99), following [26], we get the known formula:

$$I_q^{\alpha,\beta,\eta} t^\mu = \frac{\Gamma_q(\mu+1)\Gamma_q(\mu+1-\beta+\eta)}{\Gamma_q(\mu+1-\beta)\Gamma_q(\mu+1+\alpha+\eta)} t^{\mu-\beta}, \tag{102}$$

for all $t > 0$, $0 < q < 1$, $\min(\mu, \mu-\beta+\eta) > -1$.

We now state and prove the Saigo fractional q-integral inequalities which may be regarded as q-extensions of the results derived in the previous section. Here we firstly give the q-analogues of Chebyshev-type integral inequalities involving the Saigo fractional q-integral operators.

Lemma 3.4. *Let f and g be two synchronous functions on $\mathbb{T}$ and let u and v be two nonnegative functions on $\mathbb{T}$. Then we have*

$$I_{q_1}^{\alpha,\beta,\eta} u(t) I_{q_2}^{\alpha,\beta,\eta}(vfg)(t) + I_{q_2}^{\alpha,\beta,\eta} v(t) I_{q_1}^{\alpha,\beta,\eta}(ufg)(t) \geq I_{q_1}^{\alpha,\beta,\eta}(uf)(t) I_{q_2}^{\alpha,\beta,\eta}(vg)(t) + I_{q_2}^{\alpha,\beta,\eta}(vf)(t) I_{q_1}^{\alpha,\beta,\eta}(ug)(t), \tag{103}$$

for all $t > 0$, $0 < q_1, q_2 < 1$, $\alpha > \max\{0, -\beta\}$, $\beta < 1$, $\eta - \beta > -1$.

Proof. Consider

$$F_q^\star(t,\tau) = \frac{t^{-\beta-1} q^{-\eta(\alpha+\beta)}}{\Gamma_q(\alpha)} (q\tau/t; q)_{\alpha-1} \mathscr{T}_{q,\frac{q^{\alpha+1}\tau}{t}} \Big({}_2\Phi_1[q^{\alpha+\beta}, q^{-\eta}; q^\alpha; q, q] \Big) \tag{104}$$

for $\tau \in (0, t)$, $t > 0$. We note that the function $F(t,\tau)$ remains positive for all values of $\tau \in (0, t)$ $(t > 0)$ and under the conditions imposed with Lemma 3.4.

Since f and g are two synchronous functions on $\mathbb{T}$, for all $\tau > 0$ and $\rho > 0$, then the inequality (23) is satisfied. Multiplying both side of (23) by $v(\tau) F_{q_2}^\star(t,\tau)$ ($F_q^\star(t,\tau)$ defined by (104)) and integrating the resulting identity with respect to τ from 0 to t, we get

$$I_{q_2}^{\alpha,\beta,\eta}(vfg)(t) + f(\rho) g(\rho) I_{q_2}^{\alpha,\beta,\eta} v(t) \geq g(\rho) I_{q_2}^{\alpha,\beta,\eta}(vf)(t) + f(\rho) I_{q_2}^{\alpha,\beta,\eta}(vg)(t). \tag{105}$$

Multiplying both side of (105) by $u(\rho)F^{\star}_{q_1}(t,\rho)$ and integrating the resulting identity with respect to ρ from 0 to t, we obtain

$$I^{\alpha,\beta,\eta}_{q_1}u(t)I^{\alpha,\beta,\eta}_{q_2}(vfg)(t)+I^{\alpha,\beta,\eta}_{q_2}v(t)I^{\alpha,\beta,\eta}_{q_1}(ufg)(t)\geq I^{\alpha,\beta,\eta}_{q_2}(vf)(t)I^{\alpha,\beta,\eta}_{q_1}(ug)(t)+I^{\alpha,\beta,\eta}_{q_1}(uf)(t)I^{\alpha,\beta,\eta}_{q_2}(vg)(t), \tag{106}$$

which implies (103). □

Theorem 3.5. *Let f and g be two synchronous functions on $\mathbb{T}$ and let x, y and z be three nonnegative functions on $\mathbb{T}$. Then we have*

$$\begin{aligned}
&I^{\alpha,\beta,\eta}_{q_1}x(t)\Big(I^{\alpha,\beta,\eta}_{q_1}z(t)I^{\alpha,\beta,\eta}_{q_2}(yfg)(t)+2I^{\alpha,\beta,\eta}_{q_1}y(t)I^{\alpha,\beta,\eta}_{q_2}(zfg)(t)+I^{\alpha,\beta,\eta}_{q_2}z(t)I^{\alpha,\beta,\eta}_{q_1}(yfg)(t)\Big)\\
&+\Big(I^{\alpha,\beta,\eta}_{q_1}y(t)I^{\alpha,\beta,\eta}_{q_2}z(t)+I^{\alpha,\beta,\eta}_{q_2}y(t)I^{\alpha,\beta,\eta}_{q_1}z(t)\Big)I^{\alpha,\beta,\eta}_{q_1}(xfg)(t)\\
&\geq I^{\alpha,\beta,\eta}_{q_1}x(t)\Big(I^{\alpha,\beta,\eta}_{q_1}(yf)(t)I^{\alpha,\beta,\eta}_{q_2}(zg)(t)+I^{\alpha,\beta,\eta}_{q_2}(zf)(t)I^{\alpha,\beta,\eta}_{q_1}(yg)(t)\Big)+I^{\alpha,\beta,\eta}_{q_1}y(t)\Big(I^{\alpha,\beta,\eta}_{q_1}(xf)(t)I^{\alpha,\beta,\eta}_{q_2}(zg)(t)\\
&\qquad+I^{\alpha,\beta,\eta}_{q_2}(zf)(t)I^{\alpha,\beta,\eta}_{q_1}(xg)(t)\Big)+I^{\alpha,\beta,\eta}_{q_1}z(t)\Big(I^{\alpha,\beta,\eta}_{q_1}(xf)(t)I^{\alpha,\beta,\eta}_{q_2}(yg)(t)+I^{\alpha,\beta,\eta}_{q_2}(yf)(t)I^{\alpha,\beta,\eta}_{q_1}(xg)(t)\Big),
\end{aligned} \tag{107}$$

for all $t>0$, $0<q_1,q_2<1$, $\alpha>\max\{0,-\beta\}$, $\beta<1$, $\eta-\beta>-1$.

Proof. Putting $u=y$, $v=z$ and using Lemma 3.4, we can write

$$I^{\alpha,\beta,\eta}_{q_1}y(t)I^{\alpha,\beta,\eta}_{q_2}(zfg)(t)+I^{\alpha,\beta,\eta}_{q_2}z(t)I^{\alpha,\beta,\eta}_{q_1}(yfg)(t)\geq I^{\alpha,\beta,\eta}_{q_1}(yf)(t)I^{\alpha,\beta,\eta}_{q_2}(zg)(t)+I^{\alpha,\beta,\eta}_{q_2}(zf)(t)I^{\alpha,\beta,\eta}_{q_1}(yg)(t), \tag{108}$$

Multiplying both sides of (108) by $I^{\alpha,\beta,\eta}_{q_1}x(t)$, we obtain

$$\begin{aligned}
&I^{\alpha,\beta,\eta}_{q_1}x(t)\Big(I^{\alpha,\beta,\eta}_{q_1}y(t)I^{\alpha,\beta,\eta}_{q_2}(zfg)(t)+I^{\alpha,\beta,\eta}_{q_2}z(t)I^{\alpha,\beta,\eta}_{q_1}(yfg)(t)\Big)\\
&\qquad\geq I^{\alpha,\beta,\eta}_{q_1}x(t)\Big(I^{\alpha,\beta,\eta}_{q_1}(yf)(t)I^{\alpha,\beta,\eta}_{q_2}(zg)(t)+I^{\alpha,\beta,\eta}_{q_2}(zf)(t)I^{\alpha,\beta,\eta}_{q_1}(yg)(t)\Big),
\end{aligned} \tag{109}$$

Putting $u=x$, $v=z$ and using Lemma 3.4, we can write

$$I^{\alpha,\beta,\eta}_{q_1}x(t)I^{\alpha,\beta,\eta}_{q_2}(zfg)(t)+I^{\alpha,\beta,\eta}_{q_2}z(t)I^{\alpha,\beta,\eta}_{q_1}(xfg)(t)\geq I^{\alpha,\beta,\eta}_{q_1}(xf)(t)I^{\alpha,\beta,\eta}_{q_2}(zg)(t)+I^{\alpha,\beta,\eta}_{0,t}(zf)(t)I^{\alpha,\beta,\eta}_{q_1}(xg)(t), \tag{110}$$

Multiplying both sides of (110) by $I^{\alpha,\beta,\eta}_{q_1}y(t)$, we obtain

$$\begin{aligned}
&I^{\alpha,\beta,\eta}_{q_1}y(t)\Big(I^{\alpha,\beta,\eta}_{q_1}x(t)I^{\alpha,\beta,\eta}_{q_2}(zfg)(t)+I^{\alpha,\beta,\eta}_{q_2}z(t)I^{\alpha,\beta,\eta}_{q_1}(xfg)(t)\Big)\\
&\qquad\geq I^{\alpha,\beta,\eta}_{q_1}y(t)\Big(I^{\alpha,\beta,\eta}_{q_1}(xf)(t)I^{\alpha,\beta,\eta}_{q_2}(zg)(t)+I^{\alpha,\beta,\eta}_{q_2}(zf)(t)I^{\alpha,\beta,\eta}_{q_1}(xg)(t)\Big),
\end{aligned} \tag{111}$$

With the same arguments as before, we can get

$$\begin{aligned}
&I^{\alpha,\beta,\eta}_{q_1}z(t)\Big(I^{\alpha,\beta,\eta}_{q_1}x(t)I^{\alpha,\beta,\eta}_{q_2}(yfg)(t)+I^{\alpha,\beta,\eta}_{q_2}y(t)I^{\alpha,\beta,\eta}_{q_1}(xfg)(t)\Big)\\
&\qquad\geq I^{\alpha,\beta,\eta}_{q_1}z(t)\Big(I^{\alpha,\beta,\eta}_{q_1}(xf)(t)I^{\alpha,\beta,\eta}_{q_2}(yg)(t)+I^{\alpha,\beta,\eta}_{q_2}(yf)(t)I^{\alpha,\beta,\eta}_{q_1}(xg)(t)\Big),
\end{aligned} \tag{112}$$

The required inequality (107) follows on adding the inequalities (109), (111) and (112). □

Lemma 3.6. *Let f and g be two synchronous functions on $\mathbb{T}$ and let u and v be two nonnegative functions on $\mathbb{T}$. Then we have*

$$I_{q_1}^{\alpha,\beta,\eta}u(t)I_{q_2}^{\gamma,\delta,\zeta}(vfg)(t)+I_{q_2}^{\gamma,\delta,\zeta}v(t)I_{q_1}^{\alpha,\beta,\eta}(ufg)(t)\geq I_{q_1}^{\alpha,\beta,\eta}(uf)(t)I_{q_2}^{\gamma,\delta,\zeta}(vg)(t)+I_{q_2}^{\gamma,\delta,\zeta}(vf)(t)I_{q_1}^{\alpha,\beta,\eta}(ug)(t), \quad (113)$$

for all $t>0$, $0<q_1,q_2<1$, $\alpha>\max\{0,-\beta\}$, $\gamma>\max\{0,-\delta\}$, $\beta,\delta<1$, $\eta-\beta,\zeta-\delta>-1$.

Proof. Multiplying both sides of (23) by $v(\rho)G_{q_2}(t,\rho)$, where

$$G_q^{\star}(t,\rho)=\frac{t^{-\delta-1}q^{-\zeta(\gamma+\delta)}}{\Gamma_q(\gamma)}(q\rho/t;q)_{\gamma-1}\mathscr{T}_{q,\frac{q^{\gamma+1}\rho}{t}}\Big({}_2\Phi_1[q^{\gamma+\delta},q^{-\zeta};q^{\gamma};q,q]\Big) \quad (114)$$

for $\rho\in(0,t)$, $t>0$. We can see that the function $G_q^{\star}(t,\rho)$ remains positive under the conditions stated with Lemma 3.6. Integrating the resulting inequality obtained with respect to ρ from 0 to t, we have

$$f(\tau)g(\tau)I_{q_2}^{\gamma,\delta,\zeta}v(t)+I_{q_2}^{\gamma,\delta,\zeta}(vfg)(t)\geq f(\tau)I_{q_2}^{\gamma,\delta,\zeta}(vg)(t)+g(\tau)I_{q_2}^{\gamma,\delta,\zeta}(vf)(t). \quad (115)$$

Multiplying both side of (115) by $u(\tau)F_{q_1}^{\star}(t,\tau)$ (defined by (104)) and integrating the resulting identity with respect to τ from 0 to t, we obtain

$$I_{q_2}^{\gamma,\delta,\zeta}v(t)I_{q_1}^{\alpha,\beta,\eta}(ufg)(t)+I_{q_1}^{\alpha,\beta,\eta}u(t)I_{q_2}^{\gamma,\delta,\zeta}(vfg)(t)\geq I_{q_1}^{\alpha,\beta,\eta}(uf)(t)I_{q_2}^{\gamma,\delta,\zeta}(vg)(t)+I_{q_2}^{\gamma,\delta,\zeta}(vf)(t)I_{q_1}^{\alpha,\beta,\eta}(ug)(t), \quad (116)$$

which implies (113). □

Theorem 3.7. *Let f and g be two synchronous functions on $\mathbb{T}$ and let x, y and z be three nonnegative functions on $\mathbb{T}$. Then we have*

$$\begin{aligned}
&I_{q_1}^{\alpha,\beta,\eta}x(t)\Big(I_{q_1}^{\alpha,\beta,\eta}z(t)I_{q_2}^{\gamma,\delta,\zeta}(yfg)(t)+2I_{q_1}^{\alpha,\beta,\eta}y(t)I_{q_2}^{\gamma,\delta,\zeta}(zfg)(t)+I_{q_2}^{\gamma,\delta,\zeta}z(t)\times I_{q_1}^{\alpha,\beta,\eta}(yfg)(t)\Big)\\
&+\Big(I_{q_1}^{\alpha,\beta,\eta}y(t)I_{q_2}^{\gamma,\delta,\zeta}z(t)+I_{q_2}^{\gamma,\delta,\zeta}y(t)I_{q_1}^{\alpha,\beta,\eta}z(t)\Big)I_{q_1}^{\alpha,\beta,\eta}(xfg)(t)\\
&\geq I_{q_1}^{\alpha,\beta,\eta}x(t)\Big(I_{q_1}^{\alpha,\beta,\eta}(yf)(t)I_{q_2}^{\gamma,\delta,\zeta}(zg)(t)+I_{q_2}^{\gamma,\delta,\zeta}(zf)(t)I_{q_1}^{\alpha,\beta,\eta}(yg)(t)\Big)+I_{q_2}^{\gamma,\delta,\zeta}y(t)\Big(I_{q_1}^{\alpha,\beta,\eta}(xf)(t)I_{q_2}^{\gamma,\delta,\zeta}(zg)(t)\\
&\qquad+I_{q_2}^{\gamma,\delta,\zeta}(zf)(t)I_{q_1}^{\alpha,\beta,\eta}(xg)(t)\Big)+I_{q_1}^{\alpha,\beta,\eta}z(t)\Big(I_{q_1}^{\alpha,\beta,\eta}(xf)(t)I_{q_2}^{\gamma,\delta,\zeta}(yg)(t)+I_{q_2}^{\gamma,\delta,\zeta}(yf)(t)I_{q_1}^{\alpha,\beta,\eta}(xg)(t)\Big). \quad (117)
\end{aligned}$$

for all $t>0$, $0<q_1,q_2<1$, $\alpha>\max\{0,-\beta\}$, $\gamma>\max\{0,-\delta\}$, $\beta,\delta<1$, $\eta-\beta,\zeta-\delta>-1$.

Proof. Putting $u=y$, $v=z$ and using Lemma 3.6, we can write

$$I_{q_1}^{\alpha,\beta,\eta}y(t)I_{q_2}^{\gamma,\delta,\zeta}(zfg)(t)+I_{q_2}^{\gamma,\delta,\zeta}z(t)I_{q_1}^{\alpha,\beta,\eta}(yfg)(t)\geq I_{q_1}^{\alpha,\beta,\eta}(yf)(t)I_{q_2}^{\gamma,\delta,\zeta}(zg)(t)+I_{q_2}^{\gamma,\delta,\zeta}(zf)(t)I_{q_1}^{\alpha,\beta,\eta}(yg)(t), \quad (118)$$

Multiplying both sides of (118) by $I_{q_1}^{\alpha,\beta,\eta}x(t)$, we obtain

$$\begin{aligned}
&I_{q_1}^{\alpha,\beta,\eta}x(t)\Big(I_{q_1}^{\alpha,\beta,\eta}y(t)I_{q_2}^{\gamma,\delta,\zeta}(zfg)(t)+I_{q_2}^{\gamma,\delta,\zeta}z(t)I_{q_1}^{\alpha,\beta,\eta}(yfg)(t)\Big)\\
&\qquad\geq I_{q_1}^{\alpha,\beta,\eta}x(t)\Big(I_{q_1}^{\alpha,\beta,\eta}(yf)(t)I_{q_2}^{\gamma,\delta,\zeta}(zg)(t)+I_{q_2}^{\gamma,\delta,\zeta}(zf)(t)I_{q_1}^{\alpha,\beta,\eta}(yg)(t)\Big), \quad (119)
\end{aligned}$$

Putting $u=x$, $v=z$ and using Lemma 3.6, we can write

$$I_{q_1}^{\alpha,\beta,\eta}x(t)I_{q_2}^{\gamma,\delta,\zeta}(zfg)(t)+I_{q_2}^{\gamma,\delta,\zeta}z(t)I_{q_1}^{\alpha,\beta,\eta}(xfg)(t)\geq I_{q_1}^{\alpha,\beta,\eta}(xf)(t)I_{q_2}^{\gamma,\delta,\zeta}(zg)(t)+I_{q_2}^{\gamma,\delta,\zeta}(zf)(t)I_{q_1}^{\alpha,\beta,\eta}(xg)(t), \quad (120)$$

Multiplying both sides of (120) by $I_{q_1}^{\alpha,\beta,\eta}y(t)$, we obtain

$$I_{q_1}^{\alpha,\beta,\eta}y(t)\Big(I_{q_1}^{\alpha,\beta,\eta}x(t)I_{q_2}^{\gamma,\delta,\zeta}(zfg)(t)+I_{q_2}^{\gamma,\delta,\zeta}z(t)I_{q_1}^{\alpha,\beta,\eta}(xfg)(t)\Big)$$
$$\geq I_{q_1}^{\alpha,\beta,\eta}y(t)\Big(I_{q_1}^{\alpha,\beta,\eta}(xf)(t)I_{q_2}^{\gamma,\delta,\zeta}(zg)(t)+I_{q_2}^{\gamma,\delta,\zeta}(zf)(t)I_{q_1}^{\alpha,\beta,\eta}(xg)(t)\Big), \quad (121)$$

With the same arguments as before, we can get

$$I_{q_1}^{\alpha,\beta,\eta}z(t)\Big(I_{q_1}^{\alpha,\beta,\eta}x(t)I_{q_2}^{\gamma,\delta,\zeta}(yfg)(t)+I_{q_2}^{\gamma,\delta,\zeta}y(t)I_{q_1}^{\alpha,\beta,\eta}(xfg)(t)\Big)$$
$$\geq I_{q_1}^{\alpha,\beta,\eta}z(t)\Big(I_{q_1}^{\alpha,\beta,\eta}(xf)(t)I_{q_2}^{\gamma,\delta,\zeta}(yg)(t)+I_{q_2}^{\gamma,\delta,\zeta}(yf)(t)I_{q_1}^{\alpha,\beta,\eta}(xg)(t)\Big), \quad (122)$$

The required inequality (117) follows on adding the inequalities (119), (121) and (122). □

Remark 3.8. The inequalities (107) and (117) are reversed in the following cases: (*a*) The functions f and g asynchronous on $\mathbb{T}$. (*b*) The functions x, y and z are negative on $\mathbb{T}$. (*c*) Two of he functions x, y and z are positive and the third one is negative on $\mathbb{T}$.

Remark 3.9. For $\alpha=\gamma$, $\beta=\delta$, $\eta=\zeta$, Lemma 3.6 and Theorem 3.7 immediately reduce to Lemma 3.4 and Theorem 3.5, respectively. For $u(t)=v(t)=1$ and $q_1=q_2=q$, Lemma 3.4 and 3.6 immediately reduce to Theorems 4 and 5 in [25], respectively. We observe that, if we replace β by $-\alpha$, and make use of the relation (3.8) in [26], and note the following relations:

$$I_q^{\alpha,-\alpha,\eta}f(t)=I_q^{\alpha}f(t). \quad (123)$$

Furthermore, we replace β by $-\alpha$ and δ by $-\gamma$, our results reduce to Theorems 3.1 and 3.2 due to Öğünmez and Özkan [24]. If we replace β by $-\alpha$ and δ by $-\gamma$, and make use of the relation (123), then Lemma3.6 reduces to Theorems 1 and 2 in [9]. If we replace β by $-\alpha$ (and δ by $-\gamma$ additionally for Lemma 3.6 and Theorem 3.7), set $q_1=q_2=q$, and make use of the relation (123), then Lemma 3.4 and 3.6 and Theorems 3.5 and 3.7 correspond to the known results due to Yang [31].

Nextly, we present the q-analogues of Grüss-type integral inequalities involving the Saigo fractional q-integral operators. The proof of the following results are similar to that of Theorem 3.5 and 3.7 and results on Grüss-type integral inequalities involving the Saigo fractional integral operators in Section 2, therefore, we omit the further details of the proof of the following results.

Lemma 3.10. *Let f be a function satisfying the condition (7) on $\mathbb{T}$ and let x be a function on $\mathbb{T}$. Then we have*

$$I_q^{\alpha,\beta,\eta}x(t)I_q^{\alpha,\beta,\eta}(xf^2)(t)-\Big(I_q^{\alpha,\beta,\eta}(xf)(t)\Big)^2=\Big(\Phi I_q^{\alpha,\beta,\eta}x(t)-I_q^{\alpha,\beta,\eta}(xf)(t)\Big)\Big(I_q^{\alpha,\beta,\eta}(xf)(t)-\phi I_q^{\alpha,\beta,\eta}x(t)\Big)$$
$$-I_q^{\alpha,\beta,\eta}x(t)I_q^{\alpha,\beta,\eta}\Big(x(t)(\Phi-f(t))(f(t)-\phi)\Big), \quad (124)$$

for all $t>0$, $0<q<1$, $\alpha>\max\{0,-\beta\}$, $\beta<1$, $\eta-\beta>-1$.

Theorem 3.11. *Let f and g be two functions satisfying the condition (7) on $\mathbb{T}$ and let x be a nonnegative function on $\mathbb{T}$. Then we have*

$$\Big|I_q^{\alpha,\beta,\eta}x(t)I_q^{\alpha,\beta,\eta}(xfg)(t)-I_q^{\alpha,\beta,\eta}(xf)(t)I_q^{\alpha,\beta,\eta}(xg)(t)\Big|\leq\frac{1}{4}(\Phi-\phi)(\Psi-\psi)\Big(I_q^{\alpha,\beta,\eta}x(t)\Big)^2, \quad (125)$$

for all $t>0$, $0<q<1$, $\alpha>\max\{0,-\beta\}$, $\beta<1$, $\eta-\beta>-1$.

Lemma 3.12. *Let f and g be two functions on $\mathbb{T}$ and let x and y be two nonnegative functions on $\mathbb{T}$. Then we have*

$$\begin{aligned}&\Big(I_{q_1}^{\alpha,\beta,\eta}x(t)I_{q_2}^{\gamma,\delta,\zeta}(yfg)(t)+I_{q_2}^{\gamma,\delta,\zeta}y(t)I_{q_1}^{\alpha,\beta,\eta}(xfg)(t)-I_{q_1}^{\alpha,\beta,\eta}(xf)(t)I_{q_2}^{\gamma,\delta,\zeta}(yg)(t)-I_{q_2}^{\gamma,\delta,\zeta}(yf)(t)I_{q_1}^{\alpha,\beta,\eta}(xg)(t)\Big)^2\\&\le\Big(I_{q_1}^{\alpha,\beta,\eta}x(t)I_{q_2}^{\gamma,\delta,\zeta}(yf^2)(t)+I_{q_2}^{\gamma,\delta,\zeta}y(t)I_{q_1}^{\alpha,\beta,\eta}(xf^2)(t)-2I_{q_1}^{\alpha,\beta,\eta}(xf)(t)I_{q_2}^{\gamma,\delta,\zeta}(yf)(t)\Big)\\&\qquad\times\Big(I_{q_1}^{\alpha,\beta,\eta}x(t)I_{q_2}^{\gamma,\delta,\zeta}(yg^2)(t)+I_{q_2}^{\gamma,\delta,\zeta}y(t)I_{q_1}^{\alpha,\beta,\eta}(xg^2)(t)-2I_{q_1}^{\alpha,\beta,\eta}(xg)(t)I_{q_2}^{\gamma,\delta,\zeta}(yg)(t)\Big),\end{aligned}\tag{126}$$

for all $t>0$, $0<q_1,q_2<1$, $\alpha>\max\{0,-\beta\}$, $\gamma>\max\{0,-\delta\}$, $\beta,\delta<1$, $\eta-\beta,\zeta-\delta>-1$.

Lemma 3.13. *Let f be a function on $\mathbb{T}$ and let x and y be two nonnegative functions on $\mathbb{T}$. Then we have*

$$\begin{aligned}&I_{q_1}^{\alpha,\beta,\eta}x(t)I_{q_2}^{\gamma,\delta,\zeta}(yf^2)(t)+I_{q_2}^{\gamma,\delta,\zeta}y(t)I_{q_1}^{\alpha,\beta,\eta}(xf^2)(t)-2I_{q_1}^{\alpha,\beta,\eta}(xf)(t)I_{q_2}^{\gamma,\delta,\zeta}(yf)(t)\\&=\Big(\Phi I_{q_1}^{\alpha,\beta,\eta}x(t)-I_{q_1}^{\alpha,\beta,\eta}(xf)(t)\Big)\Big(I_{q_2}^{\gamma,\delta,\zeta}(yf)(t)-\phi I_{q_2}^{\gamma,\delta,\zeta}y(t)\Big)+\Big(I_{q_1}^{\alpha,\beta,\eta}(xf)(t)-\phi I_{q_1}^{\alpha,\beta,\eta}x(t)\Big)\Big(\Phi I_{q_2}^{\gamma,\delta,\zeta}y(t)-I_{q_2}^{\gamma,\delta,\zeta}(yf)(t)\Big)\\&\qquad-I_{q_1}^{\alpha,\beta,\eta}x(t)I_{q_2}^{\gamma,\delta,\zeta}\Big(y(t)(\Phi-f(t))(f(t)-\phi)\Big)-I_{q_2}^{\gamma,\delta,\zeta}y(t)I_{q_1}^{\alpha,\beta,\eta}\Big(x(t)(\Phi-f(t))(f(t)-\phi)\Big),\end{aligned}\tag{127}$$

for all $t>0$, $0<q_1,q_2<1$, $\alpha>\max\{0,-\beta\}$, $\gamma>\max\{0,-\delta\}$, $\beta,\delta<1$, $\eta-\beta,\zeta-\delta>-1$.

Theorem 3.14. *Let f and g be two functions on satisfying the condition (7) on $\mathbb{T}$ and let x and y be two nonnegative functions on $\mathbb{T}$. Then we have*

$$\begin{aligned}&\Big(I_{q_1}^{\alpha,\beta,\eta}x(t)I_{q_2}^{\gamma,\delta,\zeta}(yfg)(t)+I_{q_2}^{\gamma,\delta,\zeta}y(t)I_{q_1}^{\alpha,\beta,\eta}(xfg)(t)-I_{q_1}^{\alpha,\beta,\eta}(xf)(t)I_{q_2}^{\gamma,\delta,\zeta}(yg)(t)-I_{q_2}^{\gamma,\delta,\zeta}(yf)(t)I_{0,t}^{\alpha,\beta,\eta}(xg)(t)\Big)^2\\&\le\Big[\Big(\Phi I_{q_1}^{\alpha,\beta,\eta}x(t)-I_{q_1}^{\alpha,\beta,\eta}(xf)(t)\Big)\Big(I_{q_2}^{\gamma,\delta,\zeta}(yf)(t)-\phi I_{q_2}^{\gamma,\delta,\zeta}y(t)\Big)+\Big(I_{q_1}^{\alpha,\beta,\eta}(xf)(t)-\phi I_{q_1}^{\alpha,\beta,\eta}x(t)\Big)\Big(\Phi I_{q_2}^{\gamma,\delta,\zeta}y(t)-I_{q_2}^{\gamma,\delta,\zeta}(yf)(t)\Big)\Big]\\&\times\Big[\Big(\Psi I_{q_1}^{\alpha,\beta,\eta}x(t)-I_{q_1}^{\alpha,\beta,\eta}(xg)(t)\Big)\Big(I_{q_2}^{\gamma,\delta,\zeta}(yg)(t)-\psi I_{q_2}^{\gamma,\delta,\zeta}y(t)\Big)+\Big(I_{q_1}^{\alpha,\beta,\eta}(xg)(t)-\psi I_{q_1}^{\alpha,\beta,\eta}x(t)\Big)\Big(\Psi I_{q_2}^{\gamma,\delta,\zeta}y(t)-I_{q_2}^{\gamma,\delta,\zeta}(yg)(t)\Big)\Big],\end{aligned}\tag{128}$$

for all $t>0$, $0<q_1,q_2<1$, $\alpha>\max\{0,-\beta\}$, $\gamma>\max\{0,-\delta\}$, $\beta,\delta<1$, $\eta-\beta,\zeta-\delta>-1$.

Theorem 3.15. *Let f and g be two functions on satisfying the condition (7) on $\mathbb{T}$ and let x and y be two nonnegative functions on $\mathbb{T}$. Then we have*

$$\begin{aligned}&\Big|I_{q_1}^{\alpha,\beta,\eta}x(t)I_{q_2}^{\gamma,\delta,\zeta}(yfg)(t)+I_{q_2}^{\gamma,\delta,\zeta}y(t)I_{q_1}^{\alpha,\beta,\eta}(xfg)(t)-I_{q_1}^{\alpha,\beta,\eta}(xf)(t)I_{q_2}^{\gamma,\delta,\zeta}(yg)(t)-I_{q_2}^{\gamma,\delta,\zeta}(yf)(t)I_{q_1}^{\alpha,\beta,\eta}(xg)(t)\Big|\\&\le I_{q_1}^{\alpha,\beta,\eta}x(t)I_{q_2}^{\gamma,\delta,\zeta}y(t)(\Phi-\phi)(\Psi-\psi),\end{aligned}\tag{129}$$

for all $t>0$, $0<q_1,q_2<1$, $\alpha>\max\{0,-\beta\}$, $\gamma>\max\{0,-\delta\}$, $\beta,\delta<1$, $\eta-\beta,\zeta-\delta>-1$.

Theorem 3.16. *Let f and g be two functions on satisfying the condition (12) on $\mathbb{T}$ and let x and y be two nonnegative functions on $\mathbb{T}$. Then we have*

$$\begin{aligned}&\Big|I_{q_1}^{\alpha,\beta,\eta}x(t)I_{q_2}^{\gamma,\delta,\zeta}(yfg)(t)+I_{q_2}^{\gamma,\delta,\zeta}y(t)I_{q_1}^{\alpha,\beta,\eta}(xfg)(t)-I_{q_1}^{\alpha,\beta,\eta}(xf)(t)I_{q_2}^{\gamma,\delta,\zeta}(yg)(t)-I_{q_2}^{\gamma,\delta,\zeta}(yf)(t)I_{q_1}^{\alpha,\beta,\eta}(xg)(t)\Big|\\&\le M\Big(I_{q_1}^{\alpha,\beta,\eta}x(t)I_{q_2}^{\gamma,\delta,\zeta}(yg^2)(t)+I_{q_2}^{\gamma,\delta,\zeta}y(t)I_{q_1}^{\alpha,\beta,\eta}(xg^2)(t)-2I_{q_1}^{\alpha,\beta,\eta}(xg)(t)I_{q_2}^{\gamma,\delta,\zeta}(yg)(t)\Big),\end{aligned}\tag{130}$$

for all $t>0$, $0<q_1,q_2<1$, $\alpha>\max\{0,-\beta\}$, $\gamma>\max\{0,-\delta\}$, $\beta,\delta<1$, $\eta-\beta,\zeta-\delta>-1$.

Theorem 3.17. *Let f and g be two functions on $\mathbb{T}$ satisfying the lipschitzian condition with the constants L_1 and L_2 and let x and y be two nonnegative functions on $\mathbb{T}$. Then we have*

$$\left|I_{q_1}^{\alpha,\beta,\eta}x(t)I_{q_2}^{\gamma,\delta,\zeta}(yfg)(t)+I_{q_2}^{\gamma,\delta,\zeta}y(t)I_{q_1}^{\alpha,\beta,\eta}(xfg)(t)-I_{q_1}^{\alpha,\beta,\eta}(xf)(t)I_{q_2}^{\gamma,\delta,\zeta}(yg)(t)-I_{q_2}^{\gamma,\delta,\zeta}(yf)(t)I_{q_1}^{\alpha,\beta,\eta}(xg)(t)\right|$$
$$\leq L_1L_2\Big(I_{q_1}^{\alpha,\beta,\eta}x(t)I_{q_2}^{\gamma,\delta,\zeta}(t^2y(t))+I_{q_2}^{\gamma,\delta,\zeta}y(t)I_{q_1}^{\alpha,\beta,\eta}(t^2x(t))-2I_{q_1}^{\alpha,\beta,\eta}(tx(t))I_{q_2}^{\gamma,\delta,\zeta}(ty(t))\Big), \quad (131)$$

for all $t>0$, $0<q_1,q_2<1$, $\alpha>\max\{0,-\beta\}$, $\gamma>\max\{0,-\delta\}$, $\beta,\delta<1$, $\eta-\beta,\zeta-\delta>-1$.

Corollary 3.18. *Let f and g be two functions on $\mathbb{T}$ and let x and y be two nonnegative functions on $\mathbb{T}$. Then we have*

$$\left|I_{q_1}^{\alpha,\beta,\eta}x(t)I_{q_2}^{\gamma,\delta,\zeta}(yfg)(t)+I_{q_2}^{\gamma,\delta,\zeta}y(t)I_{q_1}^{\alpha,\beta,\eta}(xfg)(t)-I_{q_1}^{\alpha,\beta,\eta}(xf)(t)I_{q_2}^{\gamma,\delta,\zeta}(yg)(t)-I_{q_2}^{\gamma,\delta,\zeta}(yf)(t)I_{q_1}^{\alpha,\beta,\eta}(xg)(t)\right|$$
$$\leq \|D_{q_1}f\|_\infty\|D_{q_2}g\|_\infty\Big(I_{q_1}^{\alpha,\beta,\eta}x(t)I_{q_2}^{\gamma,\delta,\zeta}(t^2y(t))+I_{q_2}^{\gamma,\delta,\zeta}y(t)I_{q_1}^{\alpha,\beta,\eta}(t^2x(t))-2I_{q_1}^{\alpha,\beta,\eta}(tx(t))I_{q_2}^{\gamma,\delta,\zeta}(ty(t))\Big), \quad (132)$$

where $\|D_qh\|_\infty=\sup_{t\in\mathbb{T}}|D_qh(t)|$, for all $t>0$, $0<q_1,q_2<1$, $\alpha>\max\{0,-\beta\}$, $\gamma>\max\{0,-\delta\}$, $\beta,\delta<1$, $\eta-\beta,\zeta-\delta>-1$.

Remark 3.19. If we replace β by $-\alpha$ and δ by $-\gamma$, set $x(t)=y(t)=p(t)$ and $q_1=q_2=q$, and make use of the relation (123), then Theorems 3.11 and 3.14 correspond to the known results due to Dahmani and Benzidane [15]. Furthermore, we set $x(t)=y(t)=1$, Theorems 3.11 and 3.14 reduce to the results in Zhu et al. [32].

Remark 3.20. If we replace β by $-\alpha$ and δ by $-\gamma$, and make use of the relation (123), then Theorems 3.14–3.17 immediately reduce to the known results due to Brahim and Taf [8, 9].

We observe that, if we replace $\beta=0$, and make use of the relation (3.8) in [26], and note the following relations: $I_q^{\alpha,0,\eta}f(t)=I_q^{\alpha,\eta}f(t)$. By putting $\beta=0$ and $\delta=0$, Lemmas 3.4, 3.6, 3.10, 3.12, 3.13 and Theorems 3.5, 3.7, 3.11, 3.14–3.17 can yield the following fractional q-integral inequalities involving the Erdélyi-Kober type fractional q-integral operators.

Finally, we note that, let $q\to1^-$, and use the limit formulas

$$\lim_{q\to1^-}\frac{(q^\alpha;q)_n}{(1-q)^n}=(\alpha)_n,\quad \lim_{q\to1^-}\Gamma_q(\alpha)=\Gamma(\alpha), \quad (133)$$

the results of Section 3 then correspond to the results obtained in Section 2.

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