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Quotient Structure of Interior-closure Texture Spaces

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Abstract. In this paper, we consider quotient structure and quotient difunctions in the context of interior and closure operators on textures in the sense of Dikranjan-Giuli. The generalizations of several results concerning separation and quotient mapping are presented. It is shown that the category of interior-closure spaces and bicontinuous difunctions has a T_0 reflection. Finally, we introduce some classes of quotient difunctions such as bi-initial and bi-final difunctions between interior-closure texture spaces.

1. Introduction

Closure spaces were introduced by Cech [9] and closure operators have been used intensively in some branches of mathematics such as Topology and Algebra. In [15], a discussion was given on closure operators in the sense of Cech on texture spaces. Besides, interior-closure operators on texture spaces in the sense of Dikranjan-Giuli which give more suitable environments for different areas were discussed in [16]. In the mentioned works, there is no a priori relation between interior and closure operators. As a result, continuity and cocontinuity of difunctions was introduced by using interior and closure operators and so formed the topological category **dfICL** of interior-closure spaces and bicontinuous difunctions.

In the categorical view, some special morphisms such as quotient, initial and final maps which are based on closure operators were studied in [18].

The aim of this paper is to give quotient structure and their properties for interior-closure texture spaces. Furthermore, initial and final morphisms with respect to interior and closure operators are studied.

The theory of texture spaces is an alternative setting for fuzzy sets and therefore, many properties of Hutton algebras (known as fuzzy lattices) can be discussed in terms of textures. There is a considerable literature on this subject, and an adequate introduction to the theory and the motivation for its study may be obtained from [3–7].

Let S be a set. A texturing is a point-separating, complete, completely distributive lattice with respect to inclusion, which contains S and $\emptyset$, and for which arbitrary meets coincide with intersections, and finite joins with unions. If $\mathcal{S}$ is a texturing of S, then the pair $(S, \mathcal{S})$ is called a *texture space* or shortly, *texture* [4].

For a texture $(S, \mathcal{S})$, most properties are conveniently defined in terms of the *p-sets* $P_s = \bigcap\{A \in \mathcal{S} \mid s \in A\}$ and dually, the *q-sets*, $Q_s = \bigvee\{A \in \mathcal{S} \mid s \notin A\}$.

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Examples 1.1. (1) For any set X, $(X, \mathcal{P}(X))$ is the *discrete* texture representing the usual set structure of X. Clearly, $P_x = \{x\}$, $Q_x = X \setminus \{x\}$ for all $x \in X$.

(2) $(L, \mathcal{L})$ is a texture where $L = (0, 1]$ and $\mathcal{L} = \{(0, r] \mid r \in [0, 1]\}$. Here $P_r = Q_r = (0, r]$ for all $r \in L$.

(3) For $\mathbb{I} = [0, 1]$ define $\mathcal{I} = \{[0, t] \mid t \in [0, 1]\} \cup \{[0, t) \mid t \in [0, 1]\}$. $(\mathbb{I}, \mathcal{I})$ is a texture, which we will refer to as the *unit interval texture*. Here $P_t = [0, t]$ and $Q_t = [0, t)$ for all $t \in \mathbb{I}$,

A ditopology [3] structure on the texture $(S, \mathcal{S})$ is a pair (τ, κ) where τ contains S and $\emptyset$, and τ is closed under finite intersections and arbitrary joins, dually , κ contains $\emptyset$ and S, and κ is closed under finite unions and arbitrary intersections. The elements of τ are called open sets and the elements of κ are called closed sets.

The *closure* and the *interior* of $A \in \mathcal{S}$ under (τ, κ) are defined by the equalities

$$[A] = \bigcap\{K \in \kappa \mid A \subseteq K\}, \quad]A[= \bigvee\{G \in \tau \mid G \subseteq A\}.$$

Obviously, $A \in \tau \iff A =]A[$ and $A \in \kappa \iff A = [A]$.

Now let us recall from [15] the definition of interior-closure texture space. Let $(S, \mathcal{S})$ be a texture space. A pair $(\mathbf{int}, \mathbf{cl})$ is called a *generalized interior-closure structure (gic-structure)* where $\mathbf{int} : \mathcal{S} \to \mathcal{S}$ and $\mathbf{cl} : \mathcal{S} \to \mathcal{S}$ are two set- valued mappings on $(S, \mathcal{S})$. A gic-structure $(\mathbf{int}, \mathbf{cl})$ is called

(i) *grounded* if $\mathbf{int}(S) = S$ and $\mathbf{cl}(\emptyset) = \emptyset$.
(ii) *isotonic* if $\forall A, B \in \mathcal{S}, A \subseteq B \implies \mathbf{int}(A) \subseteq \mathbf{int}(B)$ and $\mathbf{cl}(A) \subseteq \mathbf{cl}(B)$.
(iii) *idempotent* if $\forall A \in \mathcal{S}$, $\mathbf{int}(\mathbf{int}(A)) = \mathbf{int}(A)$ and $\mathbf{cl}(\mathbf{cl}(A)) = \mathbf{cl}(A)$.
(iv) $(\mathbf{int}, \mathbf{cl})$ is called *contractive-expansive* if $\forall A \in \mathcal{S}$, $\mathbf{int}(A) \subseteq A$ and $A \subseteq \mathbf{cl}(A)$.

Note that if $(\mathbf{int}, \mathbf{cl})$ and $(\mathbf{int}', \mathbf{cl}')$ are gic-structures on $(S, \mathcal{S})$, then $(\mathbf{int}, \mathbf{cl})$ is said to be *finer* than $(\mathbf{int}', \mathbf{cl}')$, if $\mathbf{cl}(A) \subseteq \mathbf{cl}'(A)$ and $\mathbf{int}'(A) \subseteq \mathbf{int}(A)$ for each $A \in \mathcal{S}$.

A gic-structure $(\mathbf{int}, \mathbf{cl})$ on $(S, \mathcal{S})$ is called an *interior-closure texture space* or shortly, i-c space if $(\mathbf{int}, \mathbf{cl})$ is grounded, isotonic, idempotent and contractive-expansive. Then a pair $(\mathbf{int}, \mathbf{cl})$ on $(S, \mathcal{S})$ is called i-c-structure if $(S, \mathcal{S}, \mathbf{int}, \mathbf{cl})$ is an i-c space.

One of the most useful notions in the theory of texture spaces is that of difunction. A difunction is a special type of direlation [6]. Specifically, if $(S, \mathcal{S})$, $(T, \mathcal{T})$ are textures, we will denote by $\overline{P}_{(s,t)}$, $\overline{Q}_{(s,t)}$ respectively the *p*-sets and *q*-sets for the product texture $(S \times T, \mathcal{P}(S) \otimes \mathcal{T})$ [4]. Then:

1. $r \in \mathcal{P}(S) \otimes \mathcal{T}$ is called a *relation from* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$ if it satisfies

   R1 $r \not\subseteq \overline{Q}_{(s,t)}, P_{s'} \not\subseteq Q_s \implies r \not\subseteq \overline{Q}_{(s',t)}$.

   R2 $r \not\subseteq \overline{Q}_{(s,t)} \implies \exists s' \in S$ such that $P_s \not\subseteq Q_{s'}$ and $r \not\subseteq \overline{Q}_{(s',t)}$.
2. $R \in \mathcal{P}(S) \otimes \mathcal{T}$ is called a *corelation from* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$ if it satisfies

   CR1 $\overline{P}_{(s,t)} \not\subseteq R, P_s \not\subseteq Q_{s'} \implies \overline{P}_{(s',t)} \not\subseteq R$.

   CR2 $\overline{P}_{(s,t)} \not\subseteq R \implies \exists s' \in S$ such that $P_{s'} \not\subseteq Q_s$ and $\overline{P}_{(s',t)} \not\subseteq R$.
3. A pair (r, R), where r is a relation and R a corelation from $(S, \mathcal{S})$ to $(T, \mathcal{T})$, is called a *direlation from* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$.

Example 1.2. For any texture $(S, \mathcal{S})$ the identity direlation on $(S, \mathcal{S})$ is given by

$$i = \bigvee\{\overline{P}_{(s,s)} \mid s \in S\},\ I = \bigcap\{\overline{Q}_{(s,s)} \mid s \in S^\flat\}$$

Direlations are ordered by $(r_1, R_1) \sqsubseteq (r_2, R_2) \iff r_1 \subseteq r_2$ and $R_2 \subseteq R_1$, and the direlation (r, R) on $(S, \mathcal{S})$ is called *reflexive* if $(i, I) \sqsubseteq (r, R)$.

Let (r, R) be a direlation from $(S, \mathcal{S})$ to $(T, \mathcal{T})$. Then the inverse of (r, R) from $(S, \mathcal{S})$ to $(T, \mathcal{T})$ is the direlation $(r, R)^{\leftarrow} = (R^{\leftarrow}, r^{\leftarrow})$ from $(T, \mathcal{T})$ to $(S, \mathcal{S})$ given by

$$r^{\leftarrow} = \bigcap \{\overline{Q}_{(t,s)} \mid r \not\subseteq \overline{Q}_{(s,t)}\},\ R^{\leftarrow} = \bigvee \{\overline{P}_{(t,s)} \mid \overline{P}_{(s,t)} \not\subseteq R\}$$

The direlation (r, R) on $(S, \mathcal{S})$ is called symmetric if $(r, R)^{\leftarrow} = (r, R)$.

For $A \in \mathcal{S}$, the A−sections of r and R are given by [6]:

$$\begin{aligned} r^{\rightarrow}(A) &= \bigcap \{Q_t \mid \forall s,\ r \not\subseteq \overline{Q}_{(s,t)} \implies A \subseteq Q_s\} \in \mathcal{T}, \\ R^{\rightarrow}(A) &= \bigvee \{P_t \mid \forall s,\ \overline{P}_{(s,t)} \not\subseteq R \implies P_s \subseteq A\} \in \mathcal{T}. \end{aligned}$$

For $B \in \mathcal{T}$, the B−presections of r and R are given by [6]:

$$\begin{aligned} r^{\leftarrow}(B) &= \bigvee \{P_s \mid \forall t,\ r \not\subseteq \overline{Q}_{(s,t)} \implies P_t \subseteq B\} \in \mathcal{S}, \\ R^{\leftarrow}(B) &= \bigcap \{Q_s \mid \forall t,\ \overline{P}_{(s,t)} \not\subseteq R \implies B \subseteq Q_t\} \in \mathcal{S}. \end{aligned}$$

Another important concept for direlations is that of composition [6]: If (p, P): $(S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$ and (q, Q): $(S_2, \mathcal{S}_2) \to (S_3, \mathcal{S}_3)$ are direlations then their composition $(q, Q) \circ (p, P) = (q \circ p, Q \circ P)$ from $(S_1, \mathcal{S}_1)$ to $(S_3, \mathcal{S}_3)$ is given by

$$\begin{aligned} q \circ p &= \bigvee \{\overline{P}_{(s,u)} \mid \exists t \in T \text{ with } p \not\subseteq \overline{Q}_{(s,t)} \text{ and } q \not\subseteq \overline{Q}_{(t,u)}\}, \\ Q \circ P &= \bigcap \{\overline{Q}_{(s,u)} \mid \exists t \in T \text{ with } \overline{P}_{(s,t)} \not\subseteq P \text{ and } \overline{P}_{(t,u)} \not\subseteq Q\}. \end{aligned}$$

The direlation (r, R) on $(S, \mathcal{S})$ is transitive if $(r, R) \circ (r, R) \sqsubseteq (r, R)$.

Composition combines with sections and presections as one would expect.

Lemma 1.3. *Let* $(p, P) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$ *and* $(q, Q) : (S_2, \mathcal{S}_2) \to (S_3, \mathcal{S}_3)$ *be direlations. Then:*

(i) $(q \circ p)^{\rightarrow} A = q^{\rightarrow}(p^{\rightarrow} A)$ *and* $(Q \circ P)^{\rightarrow} A = Q^{\rightarrow}(P^{\rightarrow} A)\ \forall A \in \mathcal{S}_1$.
(ii) $(q \circ p)^{\leftarrow} B = p^{\leftarrow}(q^{\leftarrow} B)$ *and* $(Q \circ P)^{\leftarrow} B = P^{\leftarrow}(Q^{\leftarrow} B)\ \forall B \in \mathcal{S}_3$.

Definition 1.4 ([6]). Let (f, F) be a direlation from $(S, \mathcal{S})$ to $(T, \mathcal{T})$. Then (f, F) is called a *difunction from* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$ if it satisfies the following two conditions.

DF1 For $s, s' \in S$, $P_s \not\subseteq Q_{s'} \implies \exists t \in T$ with $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s',t)} \not\subseteq F$.
DF2 For $t, t' \in T$ and $s \in S$, $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s,t')} \not\subseteq F \implies P_{t'} \not\subseteq Q_t$.

Theorem 1.5. *Let* $(f, F) : (S, \mathcal{S}) \to (T, \mathcal{T})$ *be a difunction. Then:*

$$\begin{aligned} &\forall A \in \mathcal{S}, f^{\leftarrow}(F^{\rightarrow} A) \subseteq A \subseteq F^{\leftarrow}(f^{\rightarrow} A) \iff \\ &\forall B \in \mathcal{T}, f^{\rightarrow}(F^{\leftarrow} B) \subseteq B \subseteq F^{\rightarrow}(f^{\leftarrow} B) \iff \\ &\forall B \in \mathcal{T}, f^{\leftarrow} B = F^{\leftarrow} B \end{aligned}$$

Definition 1.6 ([6]). Let $(f, F) : (S, \mathcal{S}) \to (T, \mathcal{T})$ be a difunction. Then (f, F) is called *surjective* if it satisfies the condition

SUR. For $t, t' \in T$, $P_t \not\subseteq Q_{t'} \implies \exists s \in S$ with $f \not\subseteq \overline{Q}_{(s,t')}$ and $\overline{P}_{(s,t)} \not\subseteq F$.

Likewise, (f, F) is called *injective* if it satisfies the condition

INJ. For $s, s' \in S$ and $t \in T$, $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s',t)} \not\subseteq F \implies P_s \not\subseteq Q_{s'}$.

If (f, F) is both injective and surjective then it is called *bijective*. Note that the image and co-image are equal under bijective difunction.

The following corollary gives some basic properties of surjective and injective difunctions.

Corollary 1.7. *Let (f, F) be a difunction from $(S, \mathcal{S})$ to $(T, \mathcal{T})$.*

(1) If (f, F) is surjective then $F^{\rightarrow}(f^{\leftarrow}B) = B = f^{\rightarrow}(F^{\leftarrow}B)$ for all $B \in \mathcal{T}$.
(2) If (f, F) is injective then $F^{\leftarrow}(f^{\rightarrow}A) = A = f^{\leftarrow}(F^{\rightarrow}A)$ for all $A \in \mathcal{S}$.

Definition 1.8. Let $(S, \mathcal{S}, \mathbf{int}_S, \mathbf{cl}_S)$ and $(T, \mathcal{T}, \mathbf{int}_T, \mathbf{cl}_T)$ be i-c spaces and (f, F) be a difunction from $(S, \mathcal{S})$ to $(T, \mathcal{T})$. Then (f, F) is called

(i) *continuous* if $\forall B \in \mathcal{T},\ F^{\leftarrow}\mathbf{int}_T(B) \subseteq \mathbf{int}_S(F^{\leftarrow}B)$.
(ii) *cocontinuous* if $\forall B \in \mathcal{T},\ \mathbf{cl}_S(f^{\leftarrow}B) \subseteq f^{\leftarrow}\mathbf{cl}_T(B)$.
(iii) *bicontinuous* if it is continuous and cocontinuous.

Further (f, F) is called dihomeorphism [17] if it is bijective and, with its inverse, is bicontinuous. The next characterization of dihomeomorphism [17] is useful in this study:

Proposition 1.9. *Let $(f, F) : (S, \mathcal{S}, \mathbf{int}_S, \mathbf{cl}_S) \to (T, \mathcal{T}, \mathbf{int}_T, \mathbf{cl}_T)$ be a bijective difunction. Then the following conditions are equivalent.*
(1) *(f, F) is a dihomeomorphism.*
(2) *$A = \mathbf{int}_S(A) \iff f^{\rightarrow}A = \mathbf{int}_T(f^{\rightarrow}A)$, and $A = \mathbf{cl}_S(A) \iff F^{\rightarrow}(A) = \mathbf{cl}_T(F^{\rightarrow}(A))$, $\forall A \in \mathcal{S}$.*
(3) *$B = \mathbf{int}_T(B) \iff F^{\leftarrow}(B) = \mathbf{int}_S(F^{\leftarrow}B)$, and $B = \mathbf{cl}_T(B) \iff \mathbf{cl}_T(B) = \mathbf{cl}_T(f^{\leftarrow}B)$, $\forall B \in \mathcal{T}$.*

In general, difunctions are not directly related to ordinary (point) functions between the base sets. But, we recall from [6, Lemma 3.4] that if $(S, \mathcal{S})$, $(T, \mathcal{T})$ are textures and the point function $\varphi : S \to T$ satisfies condition

(a) $s, s' \in S, P_s \not\subseteq Q_{s'} \implies P_{\varphi(s)} \not\subseteq Q_{\varphi(s')}$

then the equalities

$$\begin{aligned} f_\varphi &= \bigvee\{\overline{P}_{(s,t)} \mid \exists u \in S \text{ satisfying } P_s \not\subseteq Q_u \text{ and } P_{\varphi(u)} \not\subseteq Q_t\}, \\ F_\varphi &= \bigcap\{\overline{Q}_{(s,t)} \mid \exists u \in S \text{ satisfying } P_u \not\subseteq Q_s \text{ and } P_t \not\subseteq Q_{\varphi(u)}\} \end{aligned} \tag{1.1}$$

define a difunction (f_φ, F_φ) from $(S, \mathcal{S})$ to $(T, \mathcal{T})$. Moreover by [6, Lemma 3.9], if φ also satisfies

(b) $s \in S,\ P_{\varphi(s)} \not\subseteq B,\ B \in \mathcal{T},\ \implies \exists s' \in S \text{ with } P_{\varphi(s')} \not\subseteq B$

then $f_\varphi^{\leftarrow}(B) = \varphi^{-1}[B] = \bigvee\{P_u \mid \varphi(u) \subseteq B\} = F_\varphi^{\leftarrow}(B)$ for all $B \in \mathcal{T}$.

2. Quotient Textures

The equivalence direlations and the quotient texture are introduced in [2]. A direlation (r, R) on $(S, \mathcal{S})$ is called an equivalence direlation if it is reflexive, symmetric and transitive. From [2, Theorem 3.5], the sense in which an equivalence direlation gives rise to a quotient texture is described in the following theorem.

Theorem 2.1. *Let (r, R) be an equivalence direlation on $(S, \mathcal{S})$. Then there exists a point equivalence relation ρ on S, a texturing $\mathcal{U}$ of the quotient set $U = S/\rho$ and surjective difunction (f, F) from $(S, \mathcal{S})$ to $(U, \mathcal{U})$ satisfying $r = F^{\leftarrow} \circ f$ and $R = f^{\leftarrow} \circ F$.*

According to this theorem:

(1) The equivalence point relation ρ on S is given by

$$s\rho t \iff r \not\subseteq \overline{Q}_{(s,v)} \forall v \text{ with } P_t \not\subseteq Q_v,\ \overline{P}_{(u,t)} \not\subseteq R\ \forall u \text{ with } P_s \not\subseteq Q_u.$$

(2) For all $s,t \in S$, $s\rho t \iff P_s \subseteq r^{\rightarrow}(P_t)$ and $P_t \subseteq r^{\rightarrow}(P_s)$.

(3) For $A \in \mathcal{S}$, $r^{\rightarrow}A = A \iff R^{\rightarrow}A = A \iff r^{\rightarrow}A = R^{\rightarrow}A$. It is denoted by $\mathcal{R}$ the family of all $A \in \mathcal{S}$ satisfying these equivalent conditions. The elements of $\mathcal{R}$ are saturated with respect to ρ.

(4) Let φ be the canonical quotient mapping $\varphi : S \to U$, $s \to \bar{s}$. By [2, Lemma 3.8], $\mathcal{U} = \{A \mid \varphi^{-1}[A] \in \mathcal{R}\}$ is a texturing of U. The mapping φ satisfies the conditions (a) and (b) mentioned earlier. It follows that φ gives rise to a difunction $(f,F) = (f_\varphi, F_\varphi) : (S,\mathcal{S}) \to (U,\mathcal{U})$, which is called the *canonical quotient difunction*. Since φ is onto, (f,F) is surjective. Moreover, for $\forall B \in \mathcal{U}$ and $\forall A \in \mathcal{R}$,

$$f^{\leftarrow}(B) = \varphi^{-1}[B] = F^{\leftarrow}(B), \quad f^{\rightarrow}(A) = \varphi[A] = F^{\rightarrow}(A) \tag{2.1}$$

Now let us give the following results [2] which will be useful in this study:

Theorem 2.2. *Let $(g,G) : (S,\mathcal{S}) \to (T,\mathcal{T})$ be a difunction. Then (r,R) defined by $r = G^{\leftarrow} \circ g$ and $R = g^{\leftarrow} \circ G$ is an equivalence direlation on $(S,\mathcal{S})$. Moreover if $(U,\mathcal{U})$ is the quotient texture associated with (r,R) and $(f,F) : (S,\mathcal{S}) \to (U,\mathcal{U})$ is the canonical quotient difunction as in mentioned earlier then $(h,H) = (g,G) \circ (f,F)^{\leftarrow}$ is an injective difunction. Finally, if (g,G) is surjective then (h,H) is bijective.*

Theorem 2.3. *Let (r_k,R_k) be an equivalence direlation on $(S_k,\mathcal{S}_k)$ and $(U_k,\mathcal{U}_k)$ the corresponding quotient texture for $k = 1,2$. If $(g,G) : (S_1,\mathcal{S}_1) \to (S_2,\mathcal{S}_2)$ is a difunction which is compatible in the sense that $A \in \mathcal{R}_2 \implies g^{\leftarrow}A \in \mathcal{R}_1$, then there exists a difunction $(\bar{g},\bar{G}) : (U_1,\mathcal{U}_1) \to (U_2,\mathcal{U}_2)$ such that $(f_2,F_2) \circ (g,G) = (\bar{g},\bar{G}) \circ (f_1,F_1)$, where $(f_k,F_k) : (S_k,\mathcal{S}_k) \to (U_k,\mathcal{U}_k)$ are canonical quotient difunctions in the sense of (2.1) for $k = 1,2$.*

Proposition 2.4. *Let $(S,\mathcal{S})$ be a texture and $\mathcal{B} \subseteq \mathcal{S}$. We set*

$$\begin{aligned} r &= \bigvee\{\overline{P}_{(s,t)} \mid \exists P_s \not\subseteq Q_u \textit{with} \quad P_t \subseteq B \textit{ or } \quad B \subseteq Q_u, \forall B \in \mathcal{B}\} \\ R &= \bigcap\{\overline{Q}_{(s,t)} \mid \exists P_v \not\subseteq Q_s \textit{with} \quad P_v \subseteq B \textit{ or } \quad B \subseteq Q_t, \forall B \in \mathcal{B}\} \end{aligned}$$

Then (r,R) is an equivalence direlation on $(S,\mathcal{S})$. Moreover, $\mathcal{B}$ generates the set $\mathcal{R}$ for (r,R) in the sense that $\mathcal{B} \subseteq \mathcal{R}$ and every element of $\mathcal{R}$ can be written as an intersection of joins of elements of $\mathcal{B}$.

3. Interior-closure Quotient Space

Textures offer a convenient setting for the investigation of complement-free concepts in general, so much of the recent work [11–14] has proceeded independently of the fuzzy setting. In particular, the notion of interior-closure space has been introduced in [15, 16] and Section 2. The study of quotient structure in ditopological texture spaces was begun in [4] and continued in [2]. The aim of this section is to carry over the concepts and some of the results given in quotient ditopological texture spaces to the much more general framework of interior-closure texture spaces.

Firstly, let us recall [16, Lemma 1.2] that if $(S,\mathcal{S})$ is a texture and $\emptyset \in \mathcal{F} \subseteq \mathcal{S}$ is closed under arbitrary intersections and $S \in \mathcal{G} \subseteq \mathcal{S}$ is closed under arbitrary joins then $(S,\mathcal{S},\mathbf{int},\mathbf{cl})$ is a interior-closure texture space where

$$\mathbf{int}(A) = \bigvee\{H \in \mathcal{G} \mid H \subseteq A\}, \quad \mathbf{cl}(A) = \bigcap\{K \in \mathcal{F} \mid A \subseteq K\}, \forall A \in \mathcal{S}$$

Proposition 3.1. *Let $(S, \mathcal{S}, \mathbf{int}, \mathbf{cl})$ be i-c space and (r, R) be an equivalence direlation on $(S, \mathcal{S})$, $(U, \mathcal{U})$ the corresponding quotient texture modulo (r, R) and $(f, F) : (S, \mathcal{S}) \to (U, \mathcal{U})$ the canonical quotient difunction. Then the pair $(\mathbf{int}_U, \mathbf{cl}_U)$ is the finest i-c- structure on the quotient texture $(U, \mathcal{U})$ for which the difunction (f, F) is bicontinuous where for all $B \in \mathcal{U}$,*

$$\mathbf{int}_U(B) = \bigvee\{G \in \mathcal{U} \mid G \subseteq B, \mathbf{int}\, F^{\leftarrow}G = F^{\leftarrow}G\},$$
$$\mathbf{cl}_U(B) = \bigcap\{K \in \mathcal{U} \mid B \subseteq K, \mathbf{cl}\, f^{\leftarrow}K = f^{\leftarrow}K\}.$$

Proof. We consider the families $\eta = \{G \in \mathcal{U} \mid \mathbf{int}\, F^{\leftarrow}G = F^{\leftarrow}G\}$ and $\mu = \{K \in \mathcal{U} \mid \mathbf{cl}\, f^{\leftarrow}K = f^{\leftarrow}K\}$. Let $K_j \in \mu$, $j \in J$ where J is any index set. Since $(\mathbf{int}, \mathbf{cl})$ is isotonic, we have $\mathbf{cl}(\bigcap_{j\in J} K_j) \subseteq \mathbf{cl}(K_j) \subseteq K_j$ and so $\mathbf{cl}(\bigcap_{j\in J} K_j) \subseteq \bigcap_{j\in J} K_j$. Hence, the family μ is closed under arbitrary intersections. By using similar argument, it is showed that the family η is closed under arbitrary joins. Thus, $(\mathbf{int}_U, \mathbf{cl}_U)$ is the $i - c$ structure on $(U, \mathcal{U})$ by [16, Lemma 1.2]. The canonical difunction (f, F) is automatically continuous and cocontinuous.

Now suppose that $(\mathbf{int}', \mathbf{cl}')$ is an i-c operator on $(U, \mathcal{U})$ and, let $(f, F) : (S, \mathcal{S}, \mathbf{int}, \mathbf{cl}) \to (U, \mathcal{U}, \mathbf{int}', \mathbf{cl}')$ be a bicontinuous difunction. By cocontinuity of (f, F),

$$\mathbf{cl}(f^{\leftarrow}B) \subseteq f^{\leftarrow}\mathbf{cl}'(B)$$

and so

$$\mathbf{cl}(f^{\leftarrow}\mathbf{cl}'(B)) \subseteq f^{\leftarrow}\mathbf{cl}'(\mathbf{cl}'(B)) = f^{\leftarrow}\mathbf{cl}'(B), \quad \forall B \in \mathcal{U}.$$

Hence, $\mathbf{cl}'(B) \in \mu$. Because of $B \subseteq \mathbf{cl}'(B)$, we find $\mathbf{cl}(B) \subseteq \mathbf{cl}'(B)$.
Further, since (f, F) is continuous,

$$F^{\leftarrow}\mathbf{int}'(B) \subseteq \mathbf{int}(F^{\leftarrow}B)$$

and then

$$F^{\leftarrow}\mathbf{int}'(B) = F^{\leftarrow}\mathbf{int}'(\mathbf{int}'(B) \subseteq \mathbf{int}(F^{\leftarrow}\mathbf{int}'(B)), \forall B \in \mathcal{U}.$$

that is $\mathbf{int}'(B) \in \eta$. Because of $\mathbf{int}'(B) \subseteq B$ we have $\mathbf{int}'(B) \subseteq \mathbf{int}(B)$. □

Definition 3.2. With the notation above, $(U, \mathcal{U}, \mathbf{int}_U, \mathbf{cl}_U)$ is called the quotient i-c space of $(S, \mathcal{S}, \mathbf{int}, \mathbf{cl})$ modulo (r, R).

Consider the following diagram:

$$\begin{tikzcd}
(S, \mathcal{S}, \mathbf{int}_S, \mathbf{cl}_S) \arrow[r, "(f,F)"] \arrow[dr, "(g,G)\circ(f,F)"'] & (U, \mathcal{U}, \mathbf{int}_U, \mathbf{cl}_U) \arrow[d, "(g,G)"] \\
& (T, \mathcal{T}, \mathbf{int}_T, \mathbf{cl}_T)
\end{tikzcd}$$

Proposition 3.3. *Let $(f, F) : (S, \mathcal{S}, \mathbf{int}_S, \mathbf{cl}_S) \to (U, \mathcal{U}, \mathbf{int}_U, \mathbf{cl}_U)$ be canonical quotient difunction. Then arbitrary difunction $(g, G) : (U, \mathcal{U}, \mathbf{int}_U, \mathbf{cl}_U) \to (T, \mathcal{T}, \mathbf{int}_T, \mathbf{cl}_T)$ is continuous (cocontinuous, bicontinuous) if and only if $(g, G) \circ (f, F)$ is continuous (cocontinuous, bicontinuous).*

Proof. ($\Longrightarrow$) Let $B \in \mathcal{T}$. Because of $(G \circ F)^{\leftarrow}B = F^{\leftarrow}(G^{\leftarrow}B)$ and (g, G) is continuous,

$$\begin{aligned}(G \circ F)^{\leftarrow}\mathbf{int}_T(B) &= F^{\leftarrow}(G^{\leftarrow}\mathbf{int}_T(B)) \subseteq F^{\leftarrow}\mathbf{int}_U(G^{\leftarrow}B) \\ &\subseteq \mathbf{int}(F^{\leftarrow}G^{\leftarrow}(B)) \subseteq \mathbf{int}((G \circ F)^{\leftarrow}B),\end{aligned}$$

As a result, $(g,G) \circ (f,F)$ is continuous.

($\Longleftarrow$) Take $B \in \mathcal{T}$. We must show that $G^{\leftarrow}\mathbf{int}_T(B) \subseteq \mathbf{int}_U(G^{\leftarrow}B)$. Since $(\mathbf{int}_T, \mathbf{cl}_T)$ is isotonic, $\mathbf{int}_T(B) \subseteq B$ and so $G^{\leftarrow}\mathbf{int}_T(B) \subseteq G^{\leftarrow}B$. Then we may write $F^{\leftarrow}G^{\leftarrow}\mathbf{int}_T(B) \subseteq F^{\leftarrow}G^{\leftarrow}B$ and since $(g \circ f, G \circ F)$ is continuous, we have $\mathbf{int}(F^{\leftarrow}G^{\leftarrow}\mathbf{int}_T(B)) = F^{\leftarrow}G^{\leftarrow}\mathbf{int}_T(B)$. From the definition of $\mathbf{int}_U$ it is obtained $G^{\leftarrow}\mathbf{int}_T(B) \subseteq \mathbf{int}_U(G^{\leftarrow}B)$.

By using dual argument, a proof is obtained for cocontinuity and bicontinuity. $\square$

Remark 3.4. Let $(g,G) : (S,\mathcal{S},\mathbf{int}_S,\mathbf{cl}_S) \to (T,\mathcal{T},\mathbf{int}_T,\mathbf{cl}_T)$ be a bicontinuous difunction between i-c spaces. Then (r,R) defined by $r = G^{\leftarrow} \circ g$ and $R = g^{\leftarrow} \circ G$ is an equivalence direlation on $(S,\mathcal{S})$ from Theorem 2.2. Hence, (g,G) gives rise a quotient texture $(U,\mathcal{U})$ associated with (r,R) and a canonical quotient difunction (f,F) together with an injective difunction $(h,H) : (U,\mathcal{U}) \to (T,\mathcal{T})$ satisfying $(h,H) \circ (f,F) = (g,G)$.

Now, let us take the quotient i-c operator $(\mathbf{int}_U, \mathbf{cl}_U)$ on $(U,\mathcal{U})$.

$$\begin{tikzcd}
(S,\mathcal{S},\mathbf{int}_S,\mathbf{cl}_S) \arrow[r, "(g,G)"] \arrow[dr, "(f,F)"'] & (T,\mathcal{T},\mathbf{int}_T,\mathbf{cl}_T) \\
& (U,\mathcal{U},\mathbf{int}_U,\mathbf{cl}_U) \arrow[u, "(h,H)"']
\end{tikzcd}$$

Since (g,G) is bicontinuous, the difunction (h,H) is also bicontinuous by Proposition 3.3. Furthermore, if (g,G) is surjective then (h,H) is a bijection, but in general it need not to be dihomeomorpsim as the inverse may not be bicontinuous.

Now we may give the next definition:

Definition 3.5. Suppose $(S,\mathcal{S},\mathbf{int}_S,\mathbf{cl}_S)$ and $(T,\mathcal{T},\mathbf{int}_T,\mathbf{cl}_T)$ are i-c spaces and $(g,G) : (S,\mathcal{S}) \to (T,\mathcal{T})$ a difunction. Then it is called a quotient difunction if it can be expressed as the composition of a canonical quotient difunction on $(S,\mathcal{S},\mathbf{int}_S,\mathbf{cl}_S)$ and a dihomeomorphism onto $(T,\mathcal{T},\mathbf{int}_T,\mathbf{cl}_T)$.

Now consider the following families for any i-c space $(S,\mathcal{S},\mathbf{int}_S,\mathbf{cl}_S)$:

$$\mathcal{O}_{\mathbf{int}} = \{A \in \mathcal{S} \mid \mathbf{int}(A) = A\} \text{ and } \mathcal{C}_{\mathbf{cl}} = \{B \in \mathcal{S} \mid \mathbf{cl}(B) = B\}.$$

Proposition 3.6. *Let $(S,\mathcal{S},\mathbf{int}_S,\mathbf{cl}_S)$ and $(T,\mathcal{T},\mathbf{int}_T,\mathbf{cl}_T)$ be i-c spaces and $(g,G) : (S,\mathcal{S}) \to (T,\mathcal{T})$ a surjective difunction. Then the following are equivalent:*

1. *(g,G) is a quotient difunction.*
2. *$G^{\leftarrow}B \in \mathcal{O}_{\mathbf{int}_S} \Longleftrightarrow B \in \mathcal{O}_{\mathbf{int}_T}$ and $g^{\leftarrow}B \in \mathcal{C}_{\mathbf{cl}_S} \Longleftrightarrow B \in \mathcal{C}_{\mathbf{cl}_T}$*
3. *The difunction (h,H) which is in the above diagram, is a dihomemorphism.*

Proof. (1) $\Longrightarrow$ (2) Suppose that (g,G) is quotient difunction. Then we may write $(g,G) = (h,H) \circ (f,F)$, where $(f,F) : (S,\mathcal{S},\mathbf{int}_S,\mathbf{cl}_S) \to (U,\mathcal{U},\mathbf{int}_U,\mathbf{cl}_U)$ is a canonical quotient difunction and $(h,H) : (U,\mathcal{U},\mathbf{int}_U,\mathbf{cl}_U) \to (T,\mathcal{T},\mathbf{int}_T,\mathbf{cl}_T)$ a dihomeomorphism. Let $G^{\leftarrow}B \in \mathcal{O}_{\mathbf{int}_S}$. Then $\mathbf{int}_S(G^{\leftarrow}B) = G^{\leftarrow}B$. From the definiton of $(\mathbf{int}_U, \mathbf{cl}_U)$ we have:

$$G^{\leftarrow}B = F^{\leftarrow}(H^{\leftarrow}B) \in \mathcal{O}_{\mathbf{int}_S} \Longleftrightarrow H^{\leftarrow}B \in \mathcal{O}_U$$

From Proposition 1.9 (3), we have $H^{\leftarrow}B \in \mathcal{O}_{\mathbf{int}_U} \Longleftrightarrow B \in \mathcal{O}_{\mathbf{int}_T}$.
By using dual arguments, the second equivalence is obtained.

(2) $\Longrightarrow$ (3) Since (g,G) is surjective, (h,H) is bijective. We will show that Proposition 1.9 (3): Let $A \in \mathcal{U}$. Since (h,H) is injective we have:

$$G^{\leftarrow}(h^{\rightarrow}(A)) = (H \circ F)^{\leftarrow}(h^{\rightarrow}(A)) = F^{\leftarrow}(H^{\leftarrow}(h^{\rightarrow}A)) = F^{\leftarrow}A. \tag{3.1}$$

Hence,

$$A \in \mathcal{O}_{\mathbf{int}_U} \Longleftrightarrow F^{\leftarrow}A \in \mathcal{O}_{\mathbf{int}_S} \Longleftrightarrow G^{\leftarrow}(h^{\rightarrow}A) \in \mathcal{O}_{\mathbf{int}_S} \Longleftrightarrow h^{\rightarrow}A \in \mathcal{O}_{\mathbf{int}_T}$$

by (2) and (3.1). Likewise, $A \in \mathcal{C}_{\mathbf{cl}_U} \Longleftrightarrow H^{\rightarrow}A \in \mathcal{C}_{\mathbf{cl}_T}$.

(3) $\Longrightarrow$ (1) It is clear. □

Examples 3.7. (1) Let $(S, \mathcal{S}, \mathbf{int}, \mathbf{cl})$ be an i-c space and (i, I) the identity difunction on $(S, \mathcal{S})$. Note that $i^{\rightarrow}(A) = A = I^{\rightarrow}(A)$ and $i^{\leftarrow}(A) = A = I^{\leftarrow}(A)$ for all $A \in \mathcal{S}$. From Proposition 3.6, (i, I) is a quotient difunction.

(2) Let us consider the texture $(L, \mathcal{L})$ of Examples 1.1 (2). Then it can be easily seen that (r, R) is a equivalence direlation where

$$r = \{(s,t) \mid 0 < t \leq \frac{1}{2} \text{ or } \frac{1}{2} < s \leq 1\},$$
$$R = \{(s,t) \mid 0 < t \leq \frac{1}{2} \text{ or } \frac{1}{2} \leq s \leq 1\}.$$

From Theorem 2.1, the corresponding equivalence point relation ρ on L is obtained as

$$s\rho t \Longleftrightarrow 0 < s \leq \frac{1}{2},\ 0 < t \leq \frac{1}{2} \text{ or } \frac{1}{2} < s \leq 1,\ \frac{1}{2} < t \leq 1$$

There are two equivalence classes which are $[\frac{1}{2}]$ and $[1]$. Hence, we have the quotient set $L/\rho = U = \{[\frac{1}{2}], [1]\}$ and the canonical quotient mapping $\varphi : L \to U$, where

$$\varphi(s) = \begin{cases} [\frac{1}{2}] & , 0 < s \leq \frac{1}{2} \\ [1] & , \frac{1}{2} < s \leq 1 \end{cases}$$

Moreover, the quotient texturing is $\mathcal{U} = \{\emptyset, \{[\frac{1}{2}]\}, U\}$, and from (1.1), the corresponding quotient difunction is given by

$$f_\varphi = \{(s, [\frac{1}{2}]) \mid 0 < s \leq \frac{1}{2}\} \cup \{(s, [1]) \mid \frac{1}{2} < s \leq 1\},$$
$$F_\varphi = \{(s, [1]) \mid 0 < s \leq \frac{1}{2}\} \cup \{(s, [\frac{1}{2}]) \mid \frac{1}{2} < s \leq 1\}.$$

Corollary 3.8. *Let (g_j, G_j), $j = 1, 2$ be difunctions.*

(1) If (g_j, G_j), $j = 1, 2$ are quotient difunctions then the composition $(g_2, G_2) \circ (g_1, G_1) = (g_2 \circ g_1, G_2 \circ G_1)$ is also quotient difunction.

(2) If (g_j, G_j), $j = 1, 2$ are bicontinuous difunctions and the composition $(g_2, G_2) \circ (g_1, G_1)$ is a quotient difunction then (g_2, G_2) is a quotient difunction.

Proof. Since the proof is independent of structure of i-c spaces, we omit the proof which follows the same lines of [2, Corollary 4.7]. □

Corollary 3.9. *An injective quotient difunction is dihomeomorphism.*

Proof. Let (g, G) be injective quotient difunction. Since (g, G) is surjective, (g, G) is bijective difunction. Let (f, F) be quotient difunction of the quotient texture generated by (g, G) as in Remark 3.4. Then we have $r = G^{\leftarrow} \circ g = i$ and $R = g^{\leftarrow} \circ G = I$ since $(G^{\leftarrow}, g^{\leftarrow})$ is the inverse of (g, G). By Theorem 2.1, we may write

$$s\rho t \Longleftrightarrow P_s = P_t \Longleftrightarrow s = t$$

It follows that $(U, \mathcal{U}, \mathbf{int}_U, \mathbf{cl}_U)$ may be identified with $(S, \mathcal{S}, \mathbf{int}_S, \mathbf{cl}_S)$. Thus, φ becomes the identity and so $(f, F) = (i, I)$, which is a dihomeomorphism. □

Now recall from [17] that if $(S, \mathcal{S}, \mathbf{int}_S, \mathbf{cl}_S)$ and $(T, \mathcal{T}, \mathbf{int}_T, \mathbf{cl}_T)$ are gic-spaces and (g, G) is a difunction from $(S, \mathcal{S})$ to $(T, \mathcal{T})$ then (g, G) is called open (co-open) if $\forall A \in \mathcal{S}$, $g^{\to}(\mathbf{int}_S(A)) = \mathbf{int}_T(g^{\to}(A))$ ($G^{\to}(\mathbf{int}_S(A)) = \mathbf{int}_T(G^{\to}(A))$).

On the other hand, (g, G) is called closed (coclosed) if $\forall B \in \mathcal{S}$, $g^{\to}(\mathbf{cl}_S(B)) = \mathbf{cl}_T(g^{\to}(B))$ ($G^{\to}(\mathbf{cl}_S(B)) = \mathbf{cl}_T(G^{\to}(B))$).

Corollary 3.10. *Let $(f, F) : (S, \mathcal{S}, \mathbf{int}_S, \mathbf{cl}_S) \to (T, \mathcal{T}, \mathbf{int}_T, \mathbf{cl}_T)$ be a surjective bicontinuous difunction.*

(1) If (f, F) is open and closed difunction then it is quotient.
(2) If (f, F) is co-open and coclosed difunction then it is quotient.

Proof. We prove only (1), leaving the dual proof of (2) to the interested reader. We will use Proposition 3.6 (2). Suppose that (f, F) is surjective bicontinuous difunction which is open and closed. Let $B \in \mathcal{O}_{\mathbf{int}_T}$. Then $B = \mathbf{int}_T(B)$ and so $F^{\leftarrow}(B) = F^{\leftarrow}(\mathbf{int}_T(B))$. Since (f, F) is continuous, we have $F^{\leftarrow}(B) \subseteq \mathbf{int}_S(F^{\leftarrow}(B))$ and so $F^{\leftarrow}(B) \in \mathcal{O}_{\mathbf{int}_S}$.

Conversely, take $F^{\leftarrow}(B) \in \mathcal{O}_{\mathbf{int}_T}$. Then $F^{\leftarrow}(B) = \mathbf{int}_S(F^{\leftarrow}(B))$ and, by [6, Corollary 2.33(1)]

$$B = f^{\to}(F^{\leftarrow}(B)) = f^{\to}(\mathbf{int}_S(F^{\leftarrow}(B)))$$

Since (f, F) is open and by $\mathbf{int}_S(F^{\leftarrow}(B)) \in \mathcal{O}_{\mathbf{int}_S}$, we have

$$B = \mathbf{int}_T(f^{\to}(\mathbf{int}_S(F^{\leftarrow}(B)))) \subseteq \mathbf{int}_T(f^{\to}(F^{\leftarrow}(B))) = \mathbf{int}_T(B)$$

Hence, $B = \mathbf{int}_T(B)$ and so $B \in \mathcal{O}_{\mathbf{int}_T}$. By using dual arguments, it can be showed that $f^{\leftarrow}B \in \mathcal{C}_{\mathbf{cl}_S} \iff B \in \mathcal{C}_{\mathbf{cl}_T}$. □

Definition 3.11. An equivalence direlation (r, R) on an i-c space is called open(co-open, closed, coclosed) if the canonical quotient difunction is open(co-open, closed, coclosed).

Now we may give the relationship between separation axioms and equivalence direlations. Firstly let us recall the following definition of bi-T_1 i-c spaces [17].

Definition 3.12. An i-c space $(S, \mathcal{S}, \mathbf{int}, \mathbf{cl})$ is called
T_1 if $\forall A \in \mathcal{S}$ can be written as $A = \bigvee_{j \in J} K_j$, $K_j \in \mathcal{C}_{\mathbf{cl}}$,
co-T_1 if $\forall A \in \mathcal{S}$ can be written as $A = \bigcap_{j \in J} G_j$, $G_j \in \mathcal{O}_{\mathbf{int}}$,
bi-T_1 if it is both T_1 and co-T_1.

Proposition 3.13. *Let $(S, \mathcal{S}, \mathbf{int}, \mathbf{cl})$ be an i-c space and (r, R) an equivalence direlation on $(S, \mathcal{S})$. We denote by $\mathcal{R}$ the family of saturated elements of $\mathcal{S}$ modulo (r, R).*

(1) If $(S, \mathcal{S}, \mathbf{int}, \mathbf{cl})$ is T_1 and (r, R) is closed then every set in $\mathcal{R}$ can be written as a join of closed sets in $\mathcal{R}$.
(2) If $(S, \mathcal{S}, \mathbf{int}, \mathbf{cl})$ is co-T_1 and (r, R) is co-open then every set in $\mathcal{R}$ can be written as an intersection of open sets in $\mathcal{R}$.

Proof. (1) Let $(S, \mathcal{S}, \mathbf{int}, \mathbf{cl})$ be a T_1 space and (r, R) be a closed equivalence direlation. Take $A \in \mathcal{R}$. From the definition of T_1 axiom we can write

$$A = \bigvee_{j \in J} K_j, \quad K_j \in \mathcal{C}_{\mathbf{cl}},$$

and by [6, Corollary 2.12(2)] we have $r^{\to}(\bigvee_{j \in J} K_j) = \bigvee_{j \in J} r^{\to}(K_j)$.

Now we show that $B \in \mathcal{C}_{\mathbf{cl}} \implies r^{\to}(B) \in \mathcal{C}_{\mathbf{cl}}$ for all $B \in \mathcal{S}$. Firstly we note that the canonical quotient difunction (f, F) is closed and $r = F^{\leftarrow} \circ f$ since (r, R) is closed direlation. If $B \in \mathcal{C}_{\mathbf{cl}}$ then $r^{\to}(B) = F^{\leftarrow}(f^{\to}(B))$. Since (f, F) is cocontinuous, $f^{\to}(B) \in \mathcal{C}_{\mathbf{cl}}$ and $F^{\leftarrow}(f^{\to}(B)) \in \mathcal{C}_{\mathbf{cl}_U}$. By Theorem 2.1, $\mathcal{R} = \{r^{\to}(A) \mid A \in \mathcal{S}\}$ and the required result follows at once.

(2) The characteristic property of co-T_1 axiom is that every element of $\mathcal{S}$ can be written as an intersection of the elements in $\mathcal{O}_{\mathbf{int}}$. On the other hand, $B \in \mathcal{O}_{\mathbf{int}} \implies R^{\to}(B) \in \mathcal{O}_{\mathbf{int}}$ for all $B \in \mathcal{S}$. Thus, the proof is dual to (1). □

In order to discuss reflectors in the category **dfICL** we will require the next theorem. Firstly, let us recall the definition of the notion T_0 i-c space given in [17].

Definition 3.14. The i-c texture space $(S, \mathcal{S}, \mathbf{int}, \mathbf{cl})$ is T_0 if given $Q_s \not\subseteq Q_t$ there exists $B \in \mathcal{O}_{\mathbf{int}} \cup \mathcal{C}_{\mathbf{cl}}$ satisfying $P_s \not\subseteq B \not\subseteq Q_t$.

Theorem 3.15. *Let $(S, \mathcal{S}, \mathbf{int}, \mathbf{cl})$ be an i-c space, (r, R) the equivalence direlation generated by $\mathcal{B} = \mathcal{O}_{\mathbf{int}} \cup \mathcal{C}_{\mathbf{cl}}$ as in Proposition 2.4. Then the quotient i-c space $(U, \mathcal{U}, \mathbf{int}_U, \mathbf{cl}_U)$ is T_0.*

Proof. Let $\varphi : S \to U$, $s \to \bar{s}$ be surjective canonical point function corresponding to the direlation (r, R) by Theorem 2.1. Firstly we show that $\varphi[\mathcal{O}_{\mathbf{int}} \cup \mathcal{C}_{\mathbf{cl}}] = \mathcal{O}_{\mathbf{int}_U} \cup \mathcal{C}_{\mathbf{cl}_U}$: Let $M \in \mathcal{O}_{\mathbf{int}_U} \cup \mathcal{C}_{\mathbf{cl}_U}$. Then $M \in \mathcal{O}_{\mathbf{int}_U}$ or $M \in \mathcal{C}_{\mathbf{cl}_U}$. If $M \in \mathcal{O}_{\mathbf{int}_U}$ then $\mathbf{int}_U(F^{\leftarrow}M) = F^{\leftarrow}M$ and so $F^{\leftarrow}M \in \mathcal{O}_{\mathbf{int}}$ by definition of $\mathcal{O}_{\mathbf{int}_U}$. Since (f, F) is surjective we have $\varphi[F^{\leftarrow}M] = f^{\rightarrow}(F^{\leftarrow}M) = M \in \varphi[\mathcal{O}_{\mathbf{int}} \cup \mathcal{C}_{\mathbf{cl}}]$ by [6, Corollary 2.33(1)]. Likewise, it is obtained that if $M \in \mathcal{C}_{\mathbf{cl}_U}$ then $M \in \varphi[\mathcal{O}_{\mathbf{int}} \cup \mathcal{C}_{\mathbf{cl}}]$.
Conversely, take $M \in \varphi[\mathcal{O}_{\mathbf{int}} \cup \mathcal{C}_{\mathbf{cl}}]$. Then there exists $N \in \mathcal{O}_{\mathbf{int}} \cup \mathcal{C}_{\mathbf{cl}}$ such that $M = \varphi[N]$. Suppose that $N \in \mathcal{O}_{\mathbf{int}}$. By (2.1), we have $M = \varphi[N] = F^{\rightarrow}(N)$ since $N \in \mathcal{B} \subseteq \mathcal{R}$. By Theorem 2.1, $f^{\leftarrow}M = f^{\leftarrow}F^{\rightarrow}(N) = R^{\rightarrow}N = N$ and so $f^{\leftarrow}M = N = \mathbf{int}(N) = \mathbf{int}(f^{\leftarrow}M)$. Thus, it is obtained that $M \subseteq \mathbf{int}_U(M)$ and so $M \in \mathcal{O}_{\mathbf{int}_U}$. Dually, it can be easily proved that $N \in \mathcal{C}_{\mathbf{cl}}$ implies $N \in \mathcal{C}_{\mathbf{cl}_U}$.
In order to verify the property T_0 of $(\mathbf{int}_U, \mathbf{cl}_U)$, we take $s, t \in S$ with $Q_{\bar{s}} \not\subseteq Q_{\bar{t}}$. Similar to the proof of [2, Theorem 5.2] there exists $B \in \mathcal{O}_{\mathbf{int}} \cup \mathcal{C}_{\mathbf{cl}}$ such that $P_{\bar{s}} \not\subseteq \varphi[B]$ and $\varphi[B] \not\subseteq Q_{\bar{t}}$. Because of the equality $\varphi[B] \in \varphi[\mathcal{B}] = \varphi[\mathcal{O}_{\mathbf{int}} \cup \mathcal{C}_{\mathbf{cl}}]$, we have established that $(U, \mathcal{U}, \mathbf{int}_U, \mathbf{cl}_U)$ is T_0. □

In [16], the category of i-c spaces and bicontinuous difunctions was denoted by **dfICL**. Now, if in this category we restrict the objects to T_0 i-c spaces we obtain the subcategory $\mathbf{dfICL}_0$.

Theorem 3.16. *The category* $\mathbf{dfICL}_0$ *is a full reflective subcategory of* **dfICL**.

Proof. Since the morphisms of categories are same, $\mathbf{dfICL}_0$ is a full subcategory of **dfICL**. Let $(S, \mathcal{S}, \mathbf{int}, \mathbf{cl}) \in$ Ob(**dfICL**). By Theorem 3.15, the quotient texture space $(U, \mathcal{U}, \mathbf{int}_U, \mathbf{cl}_U)$ is an object in $\mathbf{dfICL}_0$. Now we prove that the canonical difunction $(f, F) : (S, \mathcal{S}, \mathbf{int}, \mathbf{cl}) \to (U, \mathcal{U}, \mathbf{int}_U, \mathbf{cl}_U)$ is a reflection [1] for the object $(S, \mathcal{S}, \mathbf{int}, \mathbf{cl})$. To establish the universal property, let $(S_0, \mathcal{S}_0, \mathbf{int}_0, \mathbf{cl}_0) \in \mathrm{Ob}(\mathbf{dfICL}_0)$ and $(g, G) : (S, \mathcal{S}, \mathbf{int}, \mathbf{cl}) \to (S_0, \mathcal{S}_0, \mathbf{int}_0, \mathbf{cl}_0)$ be a morphism in **dfICL**. We must show that the existence of a $\mathbf{dfICL}_0$ morphism (h, H) making the following diagram commutative.

$$\begin{tikzcd}
(S, \mathcal{S}, \mathbf{int}, \mathbf{cl}) \arrow[r, "{(f,F)}"] \arrow[dr, "{(g,G)}"'] & (U, \mathcal{U}, \mathbf{int}_U, \mathbf{cl}_U) \arrow[d, "{(h,H)}"] \\
 & (S_0, \mathcal{S}_0, \mathbf{int}_0, \mathbf{cl}_0)
\end{tikzcd}$$

We set $\mathcal{B}_0 = \mathbf{int}_0 \cup \mathbf{cl}_0$. Consider the corresponding equivalence direlation (r_0, R_0) which is in Proposition 2.4. By [2, Theorem 5.3], (r_0, R_0) is the identity (i_0, I_0) on $(S_0, \mathcal{S}_0)$. From the proof of Corollary 3.9 we may identify the quotient with $(S_0, \mathcal{S}_0, \mathbf{int}_0, \mathbf{cl}_0)$ and thus, canonical quotient difunction becomes (i_0, I_0). Since (g, G) is bicontinuous, we have $g^{\leftarrow}(\mathcal{O}_{\mathbf{int}_0} \cup \mathcal{C}_{\mathbf{cl}_0}) \subseteq \mathcal{O}_{\mathbf{int}} \cup \mathcal{C}_{\mathbf{cl}} = \mathcal{B} \subseteq \mathcal{R}$ and so, Theorem 2.3 gives a difunction (h, H) making the diagram

$$\begin{tikzcd}
(S, \mathcal{S}, \mathbf{int}, \mathbf{cl}) \arrow[r, "{(f,F)}"] \arrow[d, "{(g,G)}"'] & (U, \mathcal{U}, \mathbf{int}_U, \mathbf{cl}_U) \arrow[d, "{(h,H)}"] \\
(S_0, \mathcal{S}_0, \mathbf{int}_0, \mathbf{cl}_0) \arrow[r, "{(i_0,I_0)}"'] & (S_0, \mathcal{S}_0, \mathbf{int}_0, \mathbf{cl}_0)
\end{tikzcd}$$

commutative. To complete the proof we need to verify that (h, H) is a $\mathbf{dfICL}_0$ morphism. For this, (h, H) must be bicontinuous.

Take $A \in \mathcal{S}_0$. Then, by Theorem 2.3, $h^{\leftarrow}A = H^{\leftarrow}A = \varphi[g^{\leftarrow}A]$. We observe that if $A \in \mathcal{O}_{\mathbf{int}_0}$ or $A \in \mathcal{C}_{\mathbf{cl}_0}$ then $g^{\leftarrow}A \in \mathcal{O}_{\mathbf{int}} \cup \mathcal{C}_{\mathbf{cl}} = \mathcal{B} \subseteq \mathcal{R}$ so the set $g^{\leftarrow}A$ is saturated with respect to ρ. Hence,

$$F^{\leftarrow}(H^{\leftarrow}A) = f^{\leftarrow}(h^{\leftarrow}A) = \varphi^{-1}[\varphi[g^{\leftarrow}A]] = g^{\leftarrow}A = G^{\leftarrow}A$$

As a result,

$$A \in \mathcal{O}_{\mathbf{int}_0} \implies G^{\leftarrow}A \in \mathcal{O}_{\mathbf{int}} \implies F^{\leftarrow}(H^{\leftarrow}A) \in \mathcal{O}_{\mathbf{int}} \implies H^{\leftarrow}A \in \mathcal{O}_{\mathbf{int}_U}$$

which proves that (h, H) is continuous. The cocontinuity is dual and it is left to the reader. □

4. Bi-initial and Bi-final Difunctions

In the categorical setting, some special morphisms such as initial and final maps which are based on closure operators were worked in [10, 18]. In this section, we introduce initiality and finality of a difunction in a interior-closure texture space.

We begin by recalling that [15, Theorem 3.4] which gives the useful characterization of bicontinuity.

Theorem 4.1. *Let $(f, F) : (S_1, \mathcal{S}_1, \mathbf{int}_1, \mathbf{cl}_1) \to (S_2, \mathcal{S}_2, \mathbf{int}_2, \mathbf{cl}_2)$ be a difunction.*

(i) (f, F) is continuous if and only if $\mathbf{int}_2(F^{\rightarrow}(A)) \subseteq F^{\rightarrow}(\mathbf{int}_1(A))$, $\forall A \in \mathcal{S}_1$,
(ii) (f, F) is cocontinuous if and only if $f^{\rightarrow}(\mathbf{cl}_1(A)) \subseteq \mathbf{cl}_2(f^{\rightarrow}(A))$, $\forall A \in \mathcal{S}_1$.

Note 4.2. By Theorem 1.5, we have:

$$(f, F) \text{ is continuous} \iff f^{\leftarrow}(\mathbf{int}_2(F^{\rightarrow}(A))) \subseteq \mathbf{int}_1(A),$$
$$(f, F) \text{ is cocontinuous} \iff \mathbf{cl}_1(A) \subseteq F^{\leftarrow}(\mathbf{cl}_2(f^{\rightarrow}(A))).$$

Now we recall [10] that a function $f : (X, \mathbf{cl}_X) \to (Y, \mathbf{cl}_Y)$ between closure spaces is called *initial* if $f^{-1}(\mathbf{cl}_Y f(A)) = \mathbf{cl}_X(A)$, $\forall A \subseteq X$. This leads to the following concepts for a difunction between i-c texture spaces.

Definition 4.3. Let $(\mathbf{int}_j, \mathbf{cl}_j)$ be i-c structure on $(S_j, \mathcal{S}_j)$, $j = 1, 2$, and $(f, F) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$ be a difunction. Then (f, F) is called

(i) initial if $F^{\rightarrow}(\mathbf{int}_1(f^{\leftarrow}(B))) = \mathbf{int}_2(B)$, $\forall B \in \mathcal{S}_2$,
(ii) co-initial if $F^{\leftarrow}(\mathbf{cl}_2(f^{\rightarrow}(A))) = \mathbf{cl}_1(A)$, $\forall A \in \mathcal{S}_1$,
(iii) bi-initial if it is initial and co-initial.

Because of $f^{\rightarrow}(F^{\leftarrow}(B)) \subseteq B$, and $A \subseteq F^{\leftarrow}(f^{\rightarrow}(A))$, it is easily obtained that every (co)-initial difunction is (co)-continuous.

Proposition 4.4. *In the category* **dfIC***:*

(i) Every retraction is initial.
(ii) Every section is co-initial.

Proof. We prove (ii), leaving the dual proof of (i) to the interested reader. Let $(f, F) : (S_1, \mathcal{S}_1, \mathbf{int}_1, \mathbf{cl}_1) \to (S_2, \mathcal{S}_2, \mathbf{int}_2, \mathbf{cl}_2)$ be a section in the category **dfIC**. Then there exists a difunction $(g, G) : (S_2, \mathcal{S}_2, \mathbf{int}_2, \mathbf{cl}_2) \to (S_1, \mathcal{S}_1, \mathbf{int}_1, \mathbf{cl}_1)$ satisfying

$$(g, G) \circ (f, F) = (g \circ f, G \circ F) = (i_{S_1}, I_{S_1}).$$

By Theorem 1.5 and the cocontinuity of (g, G), we have:

$$\begin{aligned} F^{\leftarrow}(\mathbf{cl}_2 f^{\rightarrow}(A)) &= g^{\rightarrow}(f^{\rightarrow}(F^{\leftarrow}(\mathbf{cl}_2 f^{\rightarrow}(A)))) \\ &\subseteq g^{\rightarrow}(\mathbf{cl}_2(f^{\rightarrow}(A))) \\ &\subseteq \mathbf{cl}_1(g^{\rightarrow}(f^{\rightarrow}(A))) = \mathbf{cl}_1(A). \end{aligned}$$

Since (f, F) is cocontinuous, $\mathbf{cl}_1(A) \subseteq F^{\leftarrow}(\mathbf{cl}_2(f^{\rightarrow}(A)))$. It shows co-initiality of (f, F). □

Examples 4.5. (1) Let $(S, \mathcal{S}, \mathbf{int}, \mathbf{cl})$ be an i-c space and (i, I) the identity difunction on $(S, \mathcal{S})$. Clearly, (i, I) is an initial and co-initial difunction.

(2) Let $S = T = \{a, b\}$ and define an i-c structure $(\mathbf{int}_S, \mathbf{cl}_S)$ on $(S, \mathcal{P}(S))$ by $\mathbf{cl}_S(\emptyset) = \emptyset$, $\mathbf{cl}_S(\{a\}) = \{a\}$, $\mathbf{cl}_S(\{b\}) = \mathbf{cl}_S(S) = S$, and $\mathbf{int}_S(S) = S$, $\mathbf{int}_S(\{a\}) = \mathbf{int}_S(\emptyset) = \emptyset$, $\mathbf{int}_S(\{b\}) = \{b\}$.

On the other hand, define an i-c structure $(\mathbf{int}_T, \mathbf{cl}_T)$ on $(T, \mathcal{P}(T))$ by $\mathbf{cl}_T(\emptyset) = \emptyset$, $\mathbf{cl}_T(\{a\}) = \{a\}$, $\mathbf{cl}_T(\{b\}) = \{b\}$, $\mathbf{cl}_T(T) = T$ and $\mathbf{int}_T(T) = T$, $\mathbf{int}(\{a\}) = \{a\}$, $\mathbf{int}_T(\{b\}) = \{b\}$, $\mathbf{int}_T(\emptyset) = \emptyset$.
Now we consider a difunction from $(S, \mathcal{P}(S))$ to $(T, \mathcal{P}(T))$ where

$$f = \{(a, a), (b, a)\}, \quad F = \{(a, b), (b, b)\}.$$

It can be obviously seen that (f, F) is initial and co-initial difunction.

Now we recall [10] that a function $f : (X, \mathbf{cl}_X) \to (Y, \mathbf{cl}_Y)$ between closure spaces is called *final* if $f(\mathbf{cl}_X f^{-1}(B)) = \mathbf{cl}_Y(B)$, $\forall B \subseteq Y$. This leads to the following concepts for a difunction between i-c texture spaces.

Definition 4.6. Let $(\mathbf{int}_j, \mathbf{cl}_j)$ be i-c structure on $(S_j, \mathcal{S}_j)$, $j = 1, 2$ and $(f, F) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$ be a difunction. Then (f, F) is called

(i) final if $f^{\leftarrow}(\mathbf{int}_2(F^{\to}(A))) = \mathbf{int}_1(A)$, $\forall A \in \mathcal{S}_1$
(ii) co-final if $f^{\to}(\mathbf{cl}_1(F^{\leftarrow}(B))) = \mathbf{cl}_2(B)$, $\forall B \in \mathcal{S}_2$
(iii) bi-final if it is final and co-final.

By Note 4.2, every (co)-final difunction is (co)-continuous.

Example 4.7. Let $S = \{a, b, c\}$ and $T = \{a, b\}$. We define an i-c structure $(\mathbf{int}_S, \mathbf{cl}_S)$ on $(S, \mathcal{P}(S))$ where $\mathbf{cl}_S(\emptyset) = \emptyset$, $\mathbf{cl}_S(\{a\}) = \{a, c\}$, $\mathbf{cl}_S(\{b\}) = \{a, b\}$, $\mathbf{cl}_S(\{c\}) = \{b, c\}$, $\mathbf{cl}_S(\{a, b\}) = \mathbf{cl}_S(\{a, c\}) = \mathbf{cl}_S(\{b, c\}) = \mathbf{cl}_S(S) = S$, and $\mathbf{int}_S(\{a\}) = \mathbf{int}_S(\{b\}) = \mathbf{int}_S(\{c\}) = \mathbf{int}_S(\emptyset) = \emptyset$, $\mathbf{int}_S(\{b, c\}) = \{b\}$, $\mathbf{int}_S(\{a, c\}) = \{c\}$, $\mathbf{int}_S(\{a, b\}) = \{a\}$.

On the other hand, we define an i-c structure $(\mathbf{int}_T, \mathbf{cl}_T)$ on $(T, \mathcal{P}(T))$ where $\mathbf{cl}_T(\emptyset) = \emptyset$, $\mathbf{cl}_T(\{a\}) = \mathbf{cl}_T(\{b\}) = \mathbf{cl}_T(T) = T$, and $\mathbf{int}_T(T) = T$, $\mathbf{int}_T(\{a\}) = \mathbf{int}_T(\{b\}) = \mathbf{int}_T(\emptyset) = \emptyset$.
Now we consider a difunction from $(S, \mathcal{P}(S))$ to $(T, \mathcal{P}(T))$ where

$$f = \{(a, a), (b, a), (c, b)\}, \quad F = \{(a, b), (b, b), (c, a)\}.$$

It can be easily check that (f, F) is final and co-final difunction.

Proposition 4.8. *In the category* **dfICL***:*

(i) Every section is final.
(ii) Every retraction is co-final.

Proof. We prove (ii), leaving the dual proof of (i) to the interested reader. Let $(f, F) : (S_1, \mathcal{S}_1, \mathbf{int}_1, \mathbf{cl}_1) \to (S_2, \mathcal{S}_2, \mathbf{int}_2, \mathbf{cl}_2)$ be a retraction in the category **dfICL**. Then there exists a difunction $(g, G) : (S_2, \mathcal{S}_2, \mathbf{int}_2, \mathbf{cl}_2) \to (S_1, \mathcal{S}_1, \mathbf{int}_1, \mathbf{cl}_1)$ satisfying

$$(f, F) \circ (g, G) = (f \circ g, F \circ G) = (i_{S_2}, I_{S_2}).$$

By [6, Theorem 2.23] and the cocontinuity of (f, F),

$$f^{\to}(\mathbf{cl}_1(F^{\leftarrow}(B))) \subseteq \mathbf{cl}_2(f^{\to}(F^{\leftarrow}(B))) \subseteq \mathbf{cl}_2(B)$$

and

$$\begin{aligned} f^{\to}(\mathbf{cl}_1(F^{\leftarrow}(B))) &= f^{\to}(\mathbf{cl}_1 F^{\leftarrow}(f^{\to}(g^{\to}(B)))) \\ &\supseteq f^{\to}(\mathbf{cl}_1 g^{\to}(B)) \supseteq f^{\to}(g^{\to}(\mathbf{cl}_2(B))) \quad ((g, G) \text{ is cocontinuous}) \\ &= \mathbf{cl}_2(B) \end{aligned}$$

These inclusions show the co-finality of (f, F). □

Proposition 4.9. *Let (f, F) be a difunction from $(S, \mathcal{S}, \mathbf{int}_1, \mathbf{cl}_1)$ to $(T, \mathcal{T}, \mathbf{int}_2, \mathbf{cl}_2)$ and (g, G) be a difunction from $(T, \mathcal{T}, \mathbf{int}_2, \mathbf{cl}_2)$ to $(U, \mathcal{U}, \mathbf{int}_3, \mathbf{cl}_3)$. Then*

(i) *If (g, G) and (f, F) are (co)-initial, then $(g, G) \circ (f, F)$ is also (co)-initial.*

(ii) *If (g, G) and (f, F) are (co)-final then $(g, G) \circ (f, F)$ is also (co)-final.*

Proof. (*i*) We prove co-initiality of $(g, G) \circ (f, F)$. For $A \in \mathcal{S}$,

$$\begin{aligned}(G \circ F)^{\leftarrow}(\mathbf{cl}_3(g \circ f)^{\rightarrow}(A)) &= F^{\leftarrow}(G^{\leftarrow}(\mathbf{cl}_3(g^{\rightarrow}(f^{\rightarrow}(A))))) \\ &= F^{\leftarrow}(\mathbf{cl}_2 f^{\leftarrow}(A)) && ((g, G) \ is \ co-initial) \\ &= \mathbf{cl}_1(A) && ((f, F) \ is \ co-initial)\end{aligned}$$

Using a similar argument it is easy to see that the initiality of $(g, G) \circ (f, F)$.

(*ii*) Let us prove the finality of $(g, G) \circ (f, F)$.

$$\begin{aligned}(g \circ f)^{\leftarrow}(\mathbf{int}_3(G \circ F)^{\rightarrow}(A)) &= f^{\leftarrow}(g^{\leftarrow}(\mathbf{int}_3(G^{\rightarrow}(F^{\rightarrow}(A))))) \\ &= f^{\leftarrow}(\mathbf{int}_2 F^{\rightarrow}(A)) && ((g, G) \ is \ final) \\ &= \mathbf{int}_1(A) && ((f, F) \ is \ final)\end{aligned}$$

Dually, the co-finality of $(g, G) \circ (f, F)$ can be obtained. □

We collect some minor observations which illustrate these notions:

Remark 4.10. (**1**) If (f, F) is surjective and co-initial then it is co-final:
Since (f, F) is surjective, $f^{\rightarrow}(F^{\leftarrow}(B)) = B$ for all $B \in \mathcal{S}_2$. Hence,

$$\mathbf{cl}_2(B) = f^{\rightarrow}(F^{\leftarrow}(\mathbf{cl}_2(f^{\rightarrow}(F^{\leftarrow}(B))))) = f^{\rightarrow}(\mathbf{cl}_1(F^{\leftarrow}(B))).$$

Dually, if (f, F) is injective and initial then it is final.

(**2**) If (f, F) is injective and closed then it is co-initial:
By virtue of the fact that $f^{\rightarrow}(\mathbf{cl}_2(A)) = \mathbf{cl}_1(f^{\rightarrow}(A))$ and injectivity of (f, F), we have

$$F^{\leftarrow}(\mathbf{cl}_2(f^{\rightarrow}(A)) = F^{\leftarrow}(f^{\rightarrow}(\mathbf{cl}_1(A)) = \mathbf{cl}_1(A).$$

Dually, if (f, F) is surjective and co-open then it is initial.

(**3**) If (f, F) is surjective and closed then it is co-final: Indeed,

$$f^{\rightarrow}(\mathbf{cl}_1 F^{\leftarrow}(B)) = \mathbf{cl}_2(f^{\rightarrow}(F^{\leftarrow}(B))) = \mathbf{cl}_2(B).$$

Dually, if (f, F) is injective and co-open then it is final.

Proposition 4.11. *Let $(S_j, \mathcal{S}_j, \mathbf{int}_j, \mathbf{cl}_j)$, $j = 1, 2$ be i-c spaces and $(f, F) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$ be a difunction.*

(i) (f, F) is initial if and only if every $G \in \mathcal{O}_{\mathbf{int}_1}$ is the form $G = f^{\leftarrow}(H)$ for some $H \in \mathcal{O}_{\mathbf{int}_2}$.

(ii) (f, F) is co-initial if and only if every $A \in \mathcal{C}_{\mathbf{cl}_1}$ is the form $A = F^{\leftarrow}(B)$ for some $B \in \mathcal{C}_{\mathbf{cl}_2}$.

Proof. We prove (ii), leaving the dual proof of (i) to the interested reader.
($\Longrightarrow$:) Let (f, F) be a co-initial difunction and $A \in \mathcal{C}_{\mathbf{cl}_1}$. We set $B := \mathbf{cl}_2(f^{\rightarrow}(A))$. Then $B \in \mathcal{C}_{\mathbf{cl}_2}$. Hence,

$$F^{\leftarrow}(B) = F^{\leftarrow}(\mathbf{cl}_2(f^{\rightarrow}(A))) = \mathbf{cl}_1(A) = A,$$

as requested.

($\Longleftarrow$:) Let $A \in \mathcal{S}_1$. Then $\mathbf{cl}_1(A) \in \mathcal{C}_{\mathbf{cl}_1}$ since $\mathbf{cl}_1$ is idempotent. By assumption, there exists $B \in \mathcal{C}_{\mathbf{cl}_2}$ such that $\mathbf{cl}_1(A) = F^{\leftarrow}(B) = F^{\leftarrow}(\mathbf{cl}_2(B))$. Hence,

$$\begin{aligned} f^{\rightarrow}(A) &= f^{\rightarrow}(\mathbf{cl}_1(A)) = f^{\rightarrow}(F^{\leftarrow}(B)) \subseteq B = \mathbf{cl}_2(B) \\ &\Longrightarrow \mathbf{cl}_2(f^{\rightarrow}(A)) \subseteq \mathbf{cl}_2(f^{\rightarrow}\mathbf{cl}_1(A)) \subseteq B \\ &\Longrightarrow F^{\leftarrow}(\mathbf{cl}_2(f^{\rightarrow}(A))) \subseteq F^{\leftarrow}(B) = \mathbf{cl}_1(A). \end{aligned}$$

Because of (f,F) is cocontinuous we have $\mathbf{cl}_1(A) \subseteq F^{\leftarrow}(\mathbf{cl}_2(f^{\rightarrow}(A)))$ and so (f,F) is co-initial. □

Proposition 4.12. *Let $(S_j, \mathcal{S}_j, \mathbf{int}_j, \mathbf{cl}_j)$, $j = 1,2$ be i-c spaces and $(f,F) : (S_1,\mathcal{S}_1) \to (S_2,\mathcal{S}_2)$ be a difunction.*

(i) If (f,F) is final then $G \in \mathcal{O}_{\mathbf{int}_2}$ if and only if $F^{\leftarrow}(G) \in \mathcal{O}_{\mathbf{int}_2}$, $\forall\, G \in \mathcal{S}_2$.

(ii) If (f,F) is co-final then $B \in \mathcal{C}_{\mathbf{cl}_2}$ if and only if $f^{\leftarrow}(B) \in \mathcal{C}_{\mathbf{cl}_1}$, $\forall\, B \in \mathcal{S}_2$.

Proof. We prove (ii), leaving the dual proof of (i) to the interested reader.
($\Longrightarrow$:) Let $B \in \mathcal{C}_{\mathbf{cl}_2}$. Since (f,F) is co-final, $B = \mathbf{cl}_2(B) = f^{\rightarrow}(\mathbf{cl}_1 F^{\leftarrow}(B))$. Then we have

$$f^{\leftarrow}(B) = f^{\leftarrow}(\mathbf{cl}_2(B)) = f^{\leftarrow}(f^{\rightarrow}(\mathbf{cl}_1 F^{\leftarrow}(B))) \supseteq \mathbf{cl}_1(F^{\leftarrow}(B)) = \mathbf{cl}_1(f^{\leftarrow}(B))$$

Considering the fact that $f^{\leftarrow}(B) \subseteq \mathbf{cl}_1(f^{\leftarrow}(B))$ we have $f^{\leftarrow}(B) = \mathbf{cl}_1(f^{\leftarrow}(B))$, and so $f^{\leftarrow}(B) \in \mathcal{C}_{\mathbf{cl}_1}$.
($\Longleftarrow$:) Let $f^{\leftarrow}(B) = \mathbf{cl}_1(f^{\leftarrow}(B))$ for some $B \in \mathcal{S}_2$. Since (f,F) is co-final and $F^{\leftarrow}(B) = f^{\leftarrow}(B)$, $\mathbf{cl}_2(B) = f^{\rightarrow}(\mathbf{cl}_1(F^{\leftarrow}(B))) = f^{\rightarrow}(F^{\leftarrow}(B)) \subseteq B$. That is, $\mathbf{cl}_2(B) = B$. □

Corollary 4.13. *Every surjective bi-final difunction is quotient.*

Proof. Suppose that (f,F) is a surjective bi-final difunction from $(S,\mathcal{S},\mathbf{int}_1,\mathbf{cl}_1)$ to $(T,\mathcal{T},\mathbf{int}_2,\mathbf{cl}_2)$. By finality of (f,F), $G \in \mathcal{O}_{\mathbf{int}_2}$ if and only if $F^{\leftarrow}(G) \in \mathcal{O}_{\mathbf{int}_2}$, $\forall\, G \in \mathcal{S}_2$. On the other hand, by co-finality of (f,F), $B \in \mathcal{C}_{\mathbf{cl}_2}$ if and only if $f^{\leftarrow}(B) \in \mathcal{C}_{\mathbf{cl}_1}$, $\forall\, B \in \mathcal{S}_2$. From Proposition 3.6, (f,F) is immediately quotient difunction. □

Consider the following commutative diagram:

$$\begin{tikzcd} (S_3,\mathcal{S}_3,\mathbf{int}_3,\mathbf{cl}_3) \arrow[r, "(f',F')"] \arrow[d, "(p',P')"'] & (S_4,\mathcal{S}_4,\mathbf{int}_4,\mathbf{cl}_4) \arrow[d, "(p,P)"] \\ (S_1,\mathcal{S}_1,\mathbf{int}_1,\mathbf{cl}_1) \arrow[r, "(f,F)"'] & (S_2,\mathcal{S}_2,\mathbf{int}_2,\mathbf{cl}_2) \end{tikzcd} \quad (1)$$

Then $(p \circ f', P \circ F') = (f \circ p', F \circ P')$. Moreover, for all $A \in \mathcal{S}_4$

$$\begin{aligned} &p'^{\rightarrow}(F'^{\leftarrow}(A)) \subseteq F^{\leftarrow}(p^{\rightarrow}(A)) \\ &f'^{\rightarrow}(P'^{\leftarrow}(A)) \subseteq P^{\leftarrow}(f^{\rightarrow}(A)) \end{aligned} \tag{4.1}$$

Recall from [6] that for given a texture $(S,\mathcal{S})$, consider the category $\mathbb{S}$ whose objects are the sets $A \in \mathcal{S}$ and for which there is just one morphism $[A_1,A_2] : A_1 \to A_2$ if and only if $A_1 \subseteq A_2$, composition being defined by $[A_2,A_3] \circ [A_1,A_2] = [A_1,A_3] \iff A_1 \subseteq A_2 \subseteq A_3$. Then if $\mathbb{T}$ is the category associated with $(T,\mathcal{T})$ in the same way, and (f,F) is a difunction from $(S,\mathcal{S})$ to $(T,\mathcal{T})$, we may set $F^{\rightarrow}([A_1,A_2]) = [F^{\rightarrow}A_1, F^{\rightarrow}A_2]$ and $f^{\leftarrow}([B_1,B_2]) = [f^{\leftarrow}B_1, f^{\leftarrow}B_2]$ since sections and presections preserve inclusion and it is trivial to verify that $F^{\rightarrow}$ is a functor from $\mathbb{S}$ to $\mathbb{T}$ and $f^{\leftarrow}$ is a functor from $\mathbb{T}$ to $\mathbb{S}$. From [6, Theorem 2.25], the functor $f^{\leftarrow} = F^{\leftarrow}$ is an adjoint of the functor $f^{\rightarrow}$ and the functor $F^{\rightarrow}$ is an adjoint of the functor $f^{\leftarrow}$.

The diagram (1) satisfies the Beck-Chevalley Property (*BCP*) stated in [18] if, in the last inclusions 4.1, "$\subseteq$" can be replaced by "$=$".

Now we are interested in the pullback behaviour of these morphisms.

Theorem 4.14. *(1) If (f, F) and (p', P') are final then (f', F') is final.*

(2) If (f, F) is co-final and (p', P') co-initial, then (f', F') is co-final and (p, P) is co-initial.

(3) If (f, F) is co-open and (p', P') final, then (f', F') is co-open.

(4) If (f, F) is closed and (p', P') co-initial, then (f', F') is closed.

Proof. (1) Let $A \in \mathcal{S}_3$. We will show that $\mathbf{int}_3(A) = f'^{\leftarrow}(\mathbf{int}_4(F'^{\rightarrow}(A)))$. Firstly, we observe that

$$\begin{aligned} f'^{\leftarrow}(\mathbf{int}_4(F'^{\rightarrow}(A))) &\subseteq \mathbf{int}_3(A) && ((f', F') \text{ is continuous}) \\ &= p'^{\leftarrow}(f^{\leftarrow}(\mathbf{int}_2(F^{\rightarrow}(P'^{\rightarrow}(A))))) && ((f, F) \text{ and } (p', P') \text{ are final}) \\ &= f'^{\leftarrow}(p^{\leftarrow}(\mathbf{int}_2 P^{\rightarrow} F'^{\rightarrow}(A))) && ((1) \text{ is commutative}) \\ &\subseteq f'^{\leftarrow}(\mathbf{int}_4(F'^{\rightarrow}(A)) && ((p, P) \text{ is continuous}) \end{aligned}$$

Then $\mathbf{int}_3(A) = f'^{\leftarrow}(\mathbf{int}_4(F'^{\rightarrow}(A)))$ for all $A \in \mathcal{S}_3$, and so (f', F') is final.

(2) Take $B \in \mathcal{S}_4$. We must show that $\mathbf{cl}_4(B) = f'^{\rightarrow}(\mathbf{cl}_3(F'^{\leftarrow}(B)))$ for co-finality of (f', F'), and $\mathbf{cl}_4(B) = P^{\leftarrow}(\mathbf{cl}_2(p^{\rightarrow}(B)))$ for co-initiality of (p, P). We observe that

$$\begin{aligned} \mathbf{cl}_4(B) &\subseteq P^{\leftarrow}(\mathbf{cl}_2(p^{\rightarrow}(B)) && ((p, P) \text{ is cocontinuous}) \\ &= P^{\leftarrow}(f^{\rightarrow}(\mathbf{cl}_1(F^{\leftarrow}(p^{\rightarrow}(B))))) && ((f, F) \text{ is co-final}) \\ &\subseteq P^{\leftarrow}(f^{\rightarrow}\mathbf{cl}_1(p'^{\rightarrow}(F'^{\leftarrow}(B)))) && (BCP) \\ &= f'^{\rightarrow}(p'^{\leftarrow}\mathbf{cl}_1(p'^{\rightarrow}(F'^{\leftarrow}(B)))) && (BCP) \\ &= f'^{\rightarrow}(\mathbf{cl}_3(F'^{\leftarrow}(B)) && ((p', P') \text{ is co-initial}) \\ &\subseteq \mathbf{cl}_4(f'^{\rightarrow}(F'^{\leftarrow}(B))) && ((f', F') \text{ is co-continuous}) \\ &\subseteq \mathbf{cl}_4(B) \end{aligned}$$

These inclusion series show both cofinality of (f', F') and co-initiality of (p, P).

(3) Let $A \in \mathcal{S}_3$. We will show that $F'^{\rightarrow}(\mathbf{int}_3(A)) = \mathbf{int}_4(F'^{\rightarrow}(A))$.

$$\begin{aligned} \mathbf{int}_4(F'^{\rightarrow}(A)) &\supseteq p^{\leftarrow}(\mathbf{int}_2(P^{\rightarrow}(F'^{\rightarrow}(A))) && ((p, P) \text{ is cocontinuous}) \\ &= p^{\leftarrow}(\mathbf{int}_2(F^{\rightarrow}(P'^{\rightarrow}(A))) && ((1) \text{ is commutative}) \\ &= p^{\leftarrow}(F^{\rightarrow}(\mathbf{int}_1(P'^{\rightarrow}(A)))) && ((f, F) \text{ is co-open}) \\ &= F'^{\rightarrow}(p'^{\leftarrow}(\mathbf{int}_1(P'^{\rightarrow}(A)))) && (\text{BCP}) \\ &= F'^{\rightarrow}(\mathbf{int}_3(A)) && ((p', P') \text{ is initial}) \end{aligned}$$

On the other hand, $\mathbf{int}_4(F'^{\rightarrow}(A)) \subseteq F'^{\rightarrow}(\mathbf{int}_3(A))$ since (f', F') is continuous. Hence, (f', F') is co-open.

(4) Let $A \in \mathcal{S}_3$. We prove that $f'^{\rightarrow}(\mathbf{cl}_3(A)) = \mathbf{cl}_4(f'^{\rightarrow}(A))$. Note that

$$\begin{aligned} \mathbf{cl}_4(f'^{\rightarrow}(A)) &\subseteq P^{\leftarrow}(\mathbf{cl}_2(p^{\rightarrow}(f'^{\rightarrow}(A)))) && ((p, P) \text{ is cocontinuous}) \\ &\subseteq P^{\leftarrow}(\mathbf{cl}_2 f^{\rightarrow}(p'^{\rightarrow}(A))) && ((1) \text{ is commutative}) \\ &\subseteq P^{\leftarrow}(f^{\rightarrow}(\mathbf{cl}_1(p'^{\rightarrow}(A)))) && ((f, F) \text{ is closed}) \\ &\subseteq f'^{\rightarrow}(p'^{\leftarrow}(\mathbf{cl}_1(p'^{\rightarrow}(A)))) && (BCP) \\ &= f'^{\rightarrow}(\mathbf{cl}_3(A)) && ((p', P') \text{ is co-initial}) \end{aligned}$$

Since (f', F') is cocontinuous we have $f'^{\rightarrow}(\mathbf{cl}_3(A)) \subseteq \mathbf{cl}_4(f'^{\rightarrow}(A))$ and so $f'^{\rightarrow}(\mathbf{cl}_3(A)) = \mathbf{cl}_4(f'^{\rightarrow}(A))$. □

Theorem 4.15. *(1) If (f', F') is closed and (p, P) co-final, then (f, F) is closed.*

(2) If (f', F') is co-open and (p, P) final, then (f, F) is co-open.

Proof. We prove only (1), leaving the dual proof of (2) to the interested reader. Let $A \in \mathcal{S}_1$. Then

$$\begin{aligned}
\mathbf{cl}_2(f^{\rightarrow}(A)) &= p^{\rightarrow}(\mathbf{cl}_4(P^{\leftarrow}(f^{\rightarrow}(A)))) && (\,(p,P) \text{ is co-final}) \\
&\subseteq p^{\rightarrow}(\mathbf{cl}_4(f'^{\rightarrow}(P'^{\leftarrow}(A)))) && (\ \text{BCP}) \\
&\subseteq p^{\rightarrow}(f'^{\rightarrow}(\mathbf{cl}_3(P'^{\leftarrow}(A)))) && (\,(f',F') \text{ is closed}) \\
&\subseteq f^{\rightarrow}(p'^{\rightarrow}(\mathbf{cl}_3((P'^{\leftarrow}(A))))) && (\ (1) \text{ is commutative}) \\
&\subseteq f^{\rightarrow}(\mathbf{cl}_1(p'^{\rightarrow}(P'^{\leftarrow}(A)))) && (\,(p',P') \text{ is cocontinuous}) \\
&\subseteq f^{\rightarrow}(\mathbf{cl}_1(A))
\end{aligned}$$

On the other hand, $f^{\rightarrow}(\mathbf{cl}_1(A)) \subseteq \mathbf{cl}_2(f^{\rightarrow}(A))$ since (f,F) is cocontinuous, and so (f,F) is closed. □

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