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Two Regularized Solutions of an Ill-Posed Problem for The Elliptic Equation with Inhomogeneous Source

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Abstract. In this paper, we address a Cauchy problem for elliptic equations with inhomogeneous source data. The problem is shown to be ill-posed as the solution exhibits an unstable dependence on the given data functions. Here, we shall deal with this problem by using two different regularized methods. Moreover, convergence estimates are established under some priori assumptions on the exact solution. Some numerical examples are given to illuminate the effect of our methods.
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1. Introduction

The Cauchy problem for the elliptic equation has been extensively investigated in many practical areas. For example, some problems relating to geophysics [22], plasma physics [19], bioelectric field problems [14] are equivalent to solving the Cauchy problem for the elliptic equation. In this paper, we consider the following Cauchy problem for elliptic equation

$$\begin{cases} u_{tt} = Lu + f(x,t), (x,t) \in \Omega \times (0,1), \\ u(x,t) = 0, t \in (0,1), x \in \partial\Omega, \\ u(x,0) = \varphi(x), x \in \Omega \\ u_t(x,0) = g(x), x \in \Omega. \end{cases} \tag{1}$$

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where Ω is a connected bounded domain in $\mathbb{R}^n, n \geq 1$, $\partial\Omega$ is the boundary of Ω and $L : D(L) \subset H \to H$ denotes a linear densely defined self-adjoint operator. The functions $\varphi, g \in L^2(\Omega)$ and $f \in L^2(0, 1; L^2(\Omega))$ are given.

This problem is well-known to be severely ill-posed; i.e., a small perturbation in the given Cauchy data may cause a very large error on the solution. Therefore, it is very difficult to solve it by using classic numerical methods. In order to overcome this difficulty, the regularization methods are required [24, 25].

The Cauchy problem of an elliptic equation is well known to be ill-posed in the sense of Hadamard. There have been many studies on the homogeneous problem where $f = 0$ in Eq. (1). For instance, Eldén and Berntsson [12] used the logarithmic convexity method to obtain a stability result of Hölder type. Besides that, Alessandrini, Rondi, Rosset and Vessella [7] provided essentially optimal stability results, in wide generality and under substantially minimal assumptions. In 2008, Chu-Li Fu et al [29] used the method of quasi-reversibility and truncation method to solve the homogeneous problem. Recently, the homogeneous problem is investigated in series of articles of [13,15,20,23,29–37].

Although we have many works on the homogeneous case of the elliptic problem, the literature of the inhomogeneous case is quite scarce. The ealier work on the abstract elliptic second order equation with inhomogeneous source was introduced in [27] by R.E. Showalter (See page 469). Motivated by this reason, in the present paper, we propose two regularization methods to study Problem (1). Moreover, we establish some error estimates between the regularized solution and exact solution. Especially, the convergence of the approximate solution at $t = 1$ is also proved.

The paper is organized as follows. In Section 2, we present the first regularization method and obtain the convergence estimates. In Section 3, we use a second method to construct a stable approximation solution and give the convergence estimates. Finally, a numerical experiment will be given in Section 4.

2. Preliminaries

Through this paper, we assume that the functions $\varphi, g \in L^2(\Omega)$. Physically, φ, g can only be measured, there will be measured errors, and we would actually have as some functions $\varphi^\epsilon, g^\epsilon \in L^2(\Omega)$ for which

$$\left\|\varphi^\epsilon - \varphi\right\| \leq \epsilon, \left\|g^\epsilon - g\right\| \leq \epsilon,$$

where the constant $\epsilon > 0$ represents a bound on the measurement error, $\|.\|$ denotes the L^2-norm.

Since L is a linear densely defined self-adjoint and positive definite elliptic operator on a connected bounded domain Ω with zero Dirichlet boundary condition, the eigenvalues of L are given by

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq ... \leq \lambda_p \leq ...$$

and $\lambda_p \to \infty$ as $p \to \infty$ (See [18]). And the eigenfunctions of operator L are respectively given by $X_p \in H_0^1(\Omega)$. Here assume that $\{X_p\}_{p=1}^\infty$ is considered as an orthonormal basis of $L^2(\Omega)$

$$\begin{cases} LX_p(x) = -\lambda_p X_p(x), x \in \Omega \\ X_p(x) = 0, x \in \partial\Omega. \end{cases} \tag{2}$$

for $p = 1, 2, ...$

Applying the method of separation of variables, it is easy to obtain the solution of Problem (1) is as follows

$$u(x,t) = \sum_{p=1}^{\infty}\left[\cosh(\sqrt{\lambda_p}t)\varphi_p + \frac{\sinh(\sqrt{\lambda_p}t)}{\sqrt{\lambda_p}}g_p + \int_0^t \frac{\sinh\left(\sqrt{\lambda_p}(t-s)\right)}{\sqrt{\lambda_p}}f_p(s)ds\right]X_p(x) \tag{3}$$

where

$$g_p = \int_\Omega g(x)X_p(x)dx,\ \varphi_p = \int_\Omega \varphi(x)X_p(x)dx,\ f_p(s) = \int_\Omega f(x,s)X_p(x)dx. \tag{4}$$

3. The First Method.

We modify the exact solution u as follows

$$\begin{aligned} & u^\epsilon(x,t) \\ = & \sum_{p=1}^{\infty}\left[\left(\frac{\exp\{(\sqrt{\lambda_p}-\beta\lambda_p)t\} + \exp\{-\sqrt{\lambda_p}t\}}{2}\right)\varphi_p + \left(\frac{\exp\{(\sqrt{\lambda_p}-\beta\lambda_p)t\} - \exp\{-\sqrt{\lambda_p}t\}}{2\sqrt{\lambda_p}}\right)g_p\right]X_p(x) \\ & + \sum_{p=1}^{\infty}\left[\int_0^t\left(\frac{\exp\{(\sqrt{\lambda_p}-\beta\lambda_p)t - \sqrt{\lambda_p}s\} - \exp\{\sqrt{\lambda_p}(s-t)\}}{2\sqrt{\lambda_p}}\right)f_p(s)ds\right]X_p(x) \end{aligned} \tag{5}$$

where φ_p, g_p are defined by (4). And β is the regularization parameter which depends on ϵ. Let the function v^ϵ be defined

$$\begin{aligned} & v^\epsilon(x,t) \\ = & \sum_{p=1}^{\infty}\left[\left(\frac{\exp\{(\sqrt{\lambda_p}-\beta\lambda_p)t\} + \exp\{-\sqrt{\lambda_p}t\}}{2}\right)\varphi_p^\epsilon + \left(\frac{\exp\{(\sqrt{\lambda_p}-\beta\lambda_p)t\} - \exp\{-\sqrt{\lambda_p}t\}}{2\sqrt{\lambda_p}}\right)g_p^\epsilon\right]X_p(x) \\ & + \sum_{p=1}^{\infty}\left[\int_0^t\left(\frac{\exp\{(\sqrt{\lambda_p}-\beta\lambda_p)t - \sqrt{\lambda_p}s\} - \exp\{\sqrt{\lambda_p}(s-t)\}}{2\sqrt{\lambda_p}}\right)f_p(s)ds\right]X_p(x), \end{aligned} \tag{6}$$

where

$$g_p^\epsilon = \int_\Omega g^\epsilon(x)X_p(x)dx,\ \varphi_p^\epsilon = \int_\Omega \varphi^\epsilon(x)X_p(x)dx. \tag{7}$$

Regarding the stability of the regularized solution, we have the following result.

Theorem 3.1. *Let A_1 be positive number such that $\int_0^1 \exp\{\sqrt{\lambda_p} - 2\sqrt{\lambda_p}s\}f_p^2(s)ds \le A_1$. Let us choose $\beta = \frac{1}{4k\ln(\frac{1}{\epsilon})}$ $(0 < k < 2)$, then we have*

$$\|v^\epsilon(.,t) - u(.,t)\| \le \frac{C_1 A}{4k\ln(\frac{1}{\epsilon})} + \sqrt{C_1}\sqrt{2\epsilon^2 + 2\epsilon^{2-k}}, \tag{8}$$

for every $t \in [0,1]$ *and where*

$$A = \frac{\sqrt{\|u(.,1)\|^2 + \|u_t(.,1)\|^2 + A_1}}{(1-t)^2}, \ \ C_1 = \max(1, \frac{1}{\sqrt{\lambda_1}}).$$

Proof. **Step 1.** First, we estimate the following error

$$\|u(.,t) - u^{\epsilon}(.,t)\| \le \beta A.$$

Indeed, we have

$$u(x,t) = \sum_{p=1}^{\infty}\left[\left(\frac{e^{\sqrt{\lambda_p}t} + e^{-\sqrt{\lambda_p}t}}{2}\right)\varphi_p + \left(\frac{e^{\sqrt{\lambda_p}t} - e^{-\sqrt{\lambda_p}t}}{2\sqrt{\lambda_p}}\right)g_p + \int_0^t\left(\frac{e^{\sqrt{\lambda_p}(t-s)} - e^{\sqrt{\lambda_p}(s-t)}}{2\sqrt{\lambda_p}}\right)f_p(s)ds\right]X_p(x). \tag{9}$$

Then

$$u(x,t) - u^{\epsilon}(x,t) = \sum_{p=1}^{\infty}\left(\frac{e^{\sqrt{\lambda_p}t} - e^{(\sqrt{\lambda_p}-\beta\lambda_p)t}}{2}\right)\left(\varphi_p + \frac{g_p}{\sqrt{\lambda_p}} + \frac{1}{\sqrt{\lambda_p}}\int_0^t e^{-\sqrt{\lambda_p}s}f_p(s)ds\right)X_p(x). \tag{10}$$

Moreover, by taking derivative of u with respect to t, we have

$$u_t(x,t) = \sum_{p=1}^{\infty}\sqrt{\lambda_p}\left[\left(\frac{e^{\sqrt{\lambda_p}t} - e^{-\sqrt{\lambda_p}t}}{2}\right)\varphi_p + \left(\frac{e^{\sqrt{\lambda_p}t} + e^{-\sqrt{\lambda_p}t}}{2\sqrt{\lambda_p}}\right)g_p + \int_0^t\left(\frac{e^{\sqrt{\lambda_p}(t-s)} + e^{\sqrt{\lambda_p}(s-t)}}{2\sqrt{\lambda_p}}\right)f_p(s)ds\right]X_p(x).$$

It implies that

$$< u(x,t), X_p(x) > + \frac{1}{\sqrt{\lambda_p}} < u_t(x,t), X_p(x) >= e^{\sqrt{\lambda_p}t}\left(\varphi_p + \frac{g_p}{\sqrt{\lambda_p}} + \frac{1}{\sqrt{\lambda_p}}\int_0^t e^{-\sqrt{\lambda_p}s}f_p(s)ds\right). \tag{11}$$

Let $t = 1$, we get

$$< u(x,1), X_p(x) > + \frac{1}{\sqrt{\lambda_p}} < u_t(x,1), X_p(x) >= e^{\sqrt{\lambda_p}}\left(\varphi_p + \frac{g_p}{\sqrt{\lambda_p}} + \frac{1}{\sqrt{\lambda_p}}\int_0^1 e^{-\sqrt{\lambda_p}s}f_p(s)ds\right). \tag{12}$$

Therefore, we obtain

$$\begin{aligned}&\varphi_p + \frac{g_p}{\sqrt{\lambda_p}} + \frac{1}{\sqrt{\lambda_p}}\int_0^t e^{-\sqrt{\lambda_p}s}f_p(s)ds \\ &= e^{-\sqrt{\lambda_p}}\left(< u(x,1), X_p(x) > + \frac{1}{\sqrt{\lambda_p}} < u_t(x,1), X_p(x) > - \int_t^1 e^{\sqrt{\lambda_p}-\sqrt{\lambda_p}s}f_p(s)ds\right).\end{aligned} \tag{13}$$

Combine (10) and (13) yields

$$
\begin{aligned}
&< u(x,t) - u^{\epsilon}(x,t), X_p(x) > \\
&= \left(\frac{e^{\sqrt{\lambda_p}t} - e^{(\sqrt{\lambda_p}-\beta\lambda_p)t}}{2e^{\sqrt{\lambda_p}}}\right)\left(< u(x,1), X_p(x) > + \frac{1}{\sqrt{\lambda_p}} < u_t(x,1), X_p(x) > - \int_t^1 e^{\sqrt{\lambda_p} - \sqrt{\lambda_p} s} f_p(s)ds\right) \\
&= \frac{e^{(t-1)\sqrt{\lambda_p}}\left(1 - e^{-\beta\lambda_p t}\right)}{2}\left(< u(x,1), X_p(x) > + \frac{1}{\sqrt{\lambda_p}} < u_t(x,1), X_p(x) > - \int_t^1 e^{\sqrt{\lambda_p} - \sqrt{\lambda_p} s} f_p(s)ds\right).
\end{aligned}
\tag{14}
$$

Using the inequalities $(a+b)^2 \le 2(a^2+b^2)$ and $1 - e^{-m} \le m, \;\; m > 0$, we have

$$
\begin{aligned}
&| < u(x,t) - u^{\epsilon}(x,t), X_p(x) > |^2 \\
\le\; & \frac{1}{4} e^{2(t-1)\sqrt{\lambda_p}} \beta^2 \lambda_p^2 t^2 \left(| < u(x,1), X_p(x) > |^2 + \frac{1}{\lambda_p} | < u_t(x,1), X_p(x) > |^2 + \int_t^1 e^{2\sqrt{\lambda_p} - 2\sqrt{\lambda_p} s} f_p^2(s)ds\right).
\end{aligned}
\tag{15}
$$

It is easy to prove the inequality for $z, p > 0$

$$
\frac{\lambda_p^2}{e^{2z\sqrt{\lambda_p}}} \le \frac{4}{z^4}.
$$

Thus, for $t < 1$, put $z = 1 - t$, we get

$$
e^{2(t-1)\sqrt{\lambda_p}} \beta^2 \lambda_p^2 \le \frac{4\beta^2}{(1-t)^4}. \tag{16}
$$

Combine (15) with (16) we obtain

$$
\begin{aligned}
&| < u(x,t) - u^{\epsilon}(x,t), X_p(x) > |^2 \\
&\le \frac{\beta^2}{(1-t)^4}\left(| < u(x,1), X_p(x) > |^2 + \frac{1}{\lambda_p} | < u_t(x,1), X_p(x) > |^2 + \int_0^1 e^{2\sqrt{\lambda_p} - 2\sqrt{\lambda_p} s} f_p^2(s)ds\right).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
||u(.,t) - u^{\epsilon}(.,t)||^2 &= \sum_{p=1}^{\infty} | < u(x,t) - u^{\epsilon}(x,t), X_p(x) > |^2 \\
\le\; & \frac{\beta^2}{(1-t)^4} \sum_{p=1}^{\infty}\left(| < u(x,1), X_p(x) > |^2 + \frac{1}{\lambda_p} | < u_t(x,1), X_p(x) > |^2 + \int_0^1 e^{2\sqrt{\lambda_p} - 2\sqrt{\lambda_p} s} f_p^2(s)ds\right) \\
\le\; & \frac{\beta^2}{(1-t)^4}\left(||u(.,1)||^2 + \frac{1}{\lambda_1} ||u_t(.,1)||^2 + A_1\right).
\end{aligned}
$$

Thus we get the following estimate

$$
||u(.,t) - u^{\epsilon}(.,t)|| \le \frac{\beta}{(1-t)^2}\sqrt{||u(.,1)||^2 + \frac{1}{\lambda_1} ||u_t(.,1)||^2 + A_1}.
$$

Step 2.

Let u^ϵ, v^ϵ be defined by (5) and (6) respectively. Then we have the following estimate

$$\|v^\epsilon(.,t) - u^\epsilon(.,t)\| \le \sqrt{C_1}\epsilon\sqrt{2e^{\frac{1}{4\beta}} + 2}.$$

Indeed, from (5) and (6) we get

$$\begin{aligned}
\|u^\epsilon(.,t) - v^\epsilon(.,t)\|^2 &\le 2\sum_{p=1}^{\infty}\left[\left(\frac{\exp\{(\sqrt{\lambda_p} - \beta\lambda_p)t\} + \exp\{-\sqrt{\lambda_p}t\}}{2}\right)(\varphi_p^\epsilon - \varphi_p)\right]^2 \\
&\quad + 2\sum_{p=1}^{\infty}\left[\left(\frac{\exp\{(\sqrt{\lambda_p} - \beta\lambda_p)t\} - \exp\{-\sqrt{\lambda_p}t\}}{2\sqrt{\lambda_p}}\right)(g_p^\epsilon - g_p)\right]^2 \\
&\le 2C_1\sum_{p=1}^{\infty}\frac{\exp\{(\sqrt{\lambda_p} - \beta\lambda_p)t\} + 1}{2}\Big[(\varphi_p^\epsilon - \varphi_p)^2 + (g_p^\epsilon - g_p)^2\Big]
\end{aligned}$$

where $C_1 = \max(1, \frac{1}{\sqrt{\lambda_1}})$.

Using the inequality $\sqrt{\lambda_p} - \beta\lambda_p \le \frac{1}{4\beta}$, we have

$$\begin{aligned}
\|u^\epsilon(.,t) - v^\epsilon(.,t)\|^2 &\le C_1(e^{\frac{1}{4\beta}} + 1)\Big[(\varphi_p^\epsilon - \varphi_p)^2 + (g_p^\epsilon - g_p)^2\Big] \\
&\le C_1(e^{\frac{1}{4\beta}} + 1)2\epsilon^2.
\end{aligned}$$

Thus

$$\|v^\epsilon(.,t) - u^\epsilon(.,t)\| \le \sqrt{C_1}\sqrt{(2e^{\frac{1}{4\beta}} + 2)\epsilon^2} = \sqrt{C_1}\sqrt{2\epsilon^2 + 2\epsilon^{2-k}}.$$

By combining the estimate in Step 1 with the result in Step 2 we obtain

$$\begin{aligned}
\|v^\epsilon(.,t) - u(.,t)\| &\le \|v^\epsilon(.,t) - u^\epsilon(.,t)\| + \|u^\epsilon(.,t) - u(.,t)\| \\
&\le \frac{\beta}{(1-t)^2}\sqrt{\|u(.,1)\|^2 + \frac{1}{\lambda_1}\|u_t(.,1)\|^2 + A_1} + \sqrt{C_1}\sqrt{2\epsilon^2 + 2\epsilon^{2-k}} \\
&\le \frac{C_1}{4k(1-t)^2\ln(\frac{1}{\epsilon})}\sqrt{\|u(.,1)\|^2 + \|u_t(.,1)\|^2 + A_1} + \sqrt{C_1}\sqrt{2\epsilon^2 + 2\epsilon^{2-k}}.
\end{aligned}$$

□

Remark 3.2. 1. *If $f = 0$, the estimate* (8) *becomes*

$$\|v^\epsilon(.,t) - u(.,t)\| \le \frac{C_1}{4k(1-t)^2\ln(\frac{1}{\epsilon})}\sqrt{\|u(.,1)\|^2 + \|u_t(.,1)\|^2} + \sqrt{C_1}\sqrt{2\epsilon^2 + 2\epsilon^{2-k}}. \tag{17}$$

Note here that the order of error (17) *is the same with the one in paper [29] (See Theorem 2.3, page 483).*

2. *Observing Estimate* (17) *we realize the method used in Theorem 1 have a limitation in estimating the error for $t = 1$. This is disadvantage point of this estimate. Moreover, the condition of source function f in Theorem 1 is difficult to check. To improve this limitation, we introduce the following Theorem where a simple condition is used on the exact solution to dealing with the error for all t even with $t = 1$.*

Theorem 3.3. *Let A_2 be positive number such that $\|Lu(.,t)\|^2 + \frac{1}{\lambda_1}\|Lu_t(.,t)\|^2 \le A_2^2$ for $t \in [0,1]$. If we choose $\beta = \frac{1}{4k\ln(\frac{1}{\epsilon})}$ $(0 < k < 2)$, then we have*

$$\|v^\epsilon(.,t) - u(.,t)\| \le \frac{A_2}{4k\ln(\frac{1}{\epsilon})} + \sqrt{C_1}\sqrt{2\epsilon^2 + 2\epsilon^{2-k}} \tag{18}$$

for every $t \in [0,1]$.

Proof. Combining (10) and (12) we get

$$\begin{aligned} &< u(x,t) - u^\epsilon(x,t), X_p > \\ &= \sum_{p=1}^{\infty}\left(\frac{e^{\sqrt{\lambda_p}t} - e^{(\sqrt{\lambda_p}-\beta\lambda_p)t}}{2e^{\sqrt{\lambda_p}t}}\right)\left(< u(x,t), X_p(x) > + \frac{1}{\sqrt{\lambda_p}} < u_t(x,t), X_p(x) >\right)X_p(x) \\ &= \sum_{p=1}^{\infty}\left(\frac{1 - e^{-\beta\lambda_p t}}{2}\right)\left(< u(x,t), X_p(x) > + \frac{1}{\sqrt{\lambda_p}} < u_t(x,t), X_p(x) >\right)X_p(x). \end{aligned}$$

Using the inequalities $(a+b)^2 \le 2(a^2+b^2)$ and $1 - e^{-m} \le m$, $m > 0$, we have

$$\begin{aligned} &| < u(x,t) - u^\epsilon(x,t), X_p(x) > |^2 \\ &\quad \le \frac{1}{2}(1 - e^{-\beta\lambda_p t})^2\left(| < u(x,t), X_p(x) > |^2 + \frac{1}{\lambda_p}| < u_t(x,t), X_p(x) > |^2\right) \\ &\quad \le \beta^2\lambda_p^2 t^2\left(| < u(x,t), X_p(x) > |^2 + \frac{1}{\lambda_p}| < u_t(x,t), X_p(x) > |^2\right). \end{aligned} \tag{19}$$

Therefore, we obtain

$$\begin{aligned} \|u(.,t) - u^\epsilon(.,t)\|^2 &= \sum_{p=1}^{\infty}| < u(x,t) - u^\epsilon(x,t), X_p(x) > |^2 \\ &\le \beta^2\sum_{p=1}^{\infty}t^2\lambda_p^2\left(| < u(x,t), X_p(x) > |^2 + \frac{1}{\lambda_p}| < u_t(x,t), X_p(x) > |^2\right) \\ &\le \beta^2\left(\|Lu(.,t)\|^2 + \frac{1}{\lambda_1}\|Lu_t(.,t)\|^2\right). \end{aligned}$$

Then it follows that

$$\|u(.,t) - u^\epsilon(.,t)\| \le \beta A_2.$$

Using $\|v^\epsilon(.,t) - u^\epsilon(.,t)\| \le \sqrt{C_1}\sqrt{2\epsilon^2 + 2\epsilon^{2-k}}$, we get

$$\begin{aligned} \|u(.,t) - v^\epsilon(.,t)\| &\le \|u(.,t) - v^\epsilon(.,t)\| + \|u^\epsilon(.,t) - v^\epsilon(.,t)\| \\ &\le \beta A_2 + \sqrt{C_1}\sqrt{2\epsilon^2 + 2\epsilon^{2-k}} \\ &\le \frac{A_2}{4k\ln(\frac{1}{\epsilon})} + \sqrt{C_1}\sqrt{2\epsilon^2 + 2\epsilon^{2-k}}. \end{aligned}$$

□

4. The Second Method.

For α is the parameter regularization, we have the second regularized solution as follows

$$\begin{aligned} w^\epsilon(x,t) &= \sum_{p=1}^{\infty}\Big[\Big(\frac{\frac{1}{\alpha\sqrt{\lambda_p}+e^{-\sqrt{\lambda_p}t}} + e^{-\sqrt{\lambda_p}t}}{2}\Big)\varphi_p + \Big(\frac{\frac{1}{\alpha\sqrt{\lambda_p}+e^{-\sqrt{\lambda_p}t}} - e^{-\sqrt{\lambda_p}t}}{2\sqrt{\lambda_p}}\Big)g_p \\ &\quad + \int_0^t \frac{\frac{e^{-s\sqrt{\lambda_p}}}{\alpha\sqrt{\lambda_p}+e^{-\sqrt{\lambda_p}t}} - e^{\sqrt{\lambda_p}(s-t)}}{2\sqrt{\lambda_p}} f_p(s)ds\Big]X_p(x). \end{aligned} \tag{20}$$

and

$$\begin{aligned} W^\epsilon(x,t) &= \sum_{p=1}^{\infty}\Big[\Big(\frac{\frac{1}{\alpha\sqrt{\lambda_p}+e^{-\sqrt{\lambda_p}t}} + e^{-\sqrt{\lambda_p}t}}{2}\Big)\varphi_p^\epsilon + \Big(\frac{\frac{1}{\alpha\sqrt{\lambda_p}+e^{-\sqrt{\lambda_p}t}} - e^{-\sqrt{\lambda_p}t}}{2\sqrt{\lambda_p}}\Big)g_p^\epsilon \\ &\quad + \int_0^t \frac{\frac{e^{-s\sqrt{\lambda_p}}}{\alpha\sqrt{\lambda_p}+e^{-\sqrt{\lambda_p}t}} - e^{\sqrt{\lambda_p}(s-t)}}{2\sqrt{\lambda_p}} f_p(s)ds\Big]X_p(x). \end{aligned} \tag{21}$$

Theorem 4.1. *Let E be positive number such that $\|Lu(.,1)\|^2 + \|u_t(.,1)\|^2 \le E^2$. If we choose $\alpha = \epsilon^a$ $(a < 1)$, then*

$$\|W^\epsilon(.,t) - u(.,t)\| \le \frac{\sqrt{2}C_1}{\ln(\frac{1}{\epsilon^a})}E + 2C_1\epsilon^{1-a}, \tag{22}$$

for every $t \in [0,1]$ and where $C_1 = \max(1, \frac{1}{\sqrt{\lambda_1}})$.

Proof

First, we prove the following Lemma

Lemma 4.2. *Let $\varphi, g \in L^2(0,\pi)$. Then, we have*

$$\|W^\epsilon(.,t) - w^\epsilon(.,t)\| \le 2C_1\alpha^{-1}\epsilon.$$

Proof. We have

$$\begin{aligned} W^\epsilon(x,t) - w^\epsilon(x,t) &= \sum_{p=1}^{\infty}\left[\left(\frac{\frac{e^{\sqrt{\lambda_p}t}}{1+\alpha\sqrt{\lambda_p}e^{\sqrt{\lambda_p}t}} + e^{-\sqrt{\lambda_p}t}}{2}\right)(\varphi_p^\epsilon - \varphi_p)\right]X_p(x) \\ &+ \sum_{p=1}^{\infty}\left[\left(\frac{\frac{e^{\sqrt{\lambda_p}t}}{1+\alpha\sqrt{\lambda_p}e^{\sqrt{\lambda_p}t}} - e^{-\sqrt{\lambda_p}t}}{2\sqrt{\lambda_p}}\right)(g_p^\epsilon - g_p)\right]X_p(x). \end{aligned}$$

Since $\frac{e^{\sqrt{\lambda_p}t}}{1+\alpha\sqrt{\lambda_p}e^{\sqrt{\lambda_p}t}} \le \frac{1}{\alpha}$ $(*)$ and $e^{-\sqrt{\lambda_p}t} \le 1 \le \frac{1}{\alpha}$ we get

$$\begin{aligned} \|W^\epsilon(.,t) - w^\epsilon(.,t)\|^2 &\le 2\sum_{p=1}^{\infty}\left[\left(\frac{\frac{e^{\sqrt{\lambda_p}t}}{1+\alpha\sqrt{\lambda_p}e^{\sqrt{\lambda_p}t}} + e^{-\sqrt{\lambda_p}t}}{2}\right)(\varphi_p^\epsilon - \varphi_p)\right]^2 \\ &+ 2\sum_{p=1}^{\infty}\left[\left(\frac{\frac{e^{\sqrt{\lambda_p}t}}{1+\alpha\sqrt{\lambda_p}e^{\sqrt{\lambda_p}t}} - e^{-\sqrt{\lambda_p}t}}{2\sqrt{\lambda_p}}\right)(g_p^\epsilon - g_p)\right]^2 \\ &\le 2\sum_{p=1}^{\infty}\left[\left(\frac{\frac{1}{\alpha}+\frac{1}{\alpha}}{2}\right)(\varphi_p^\epsilon - \varphi_p)\right]^2 \\ &+ 2\sum_{p=1}^{\infty}\left[\left(\frac{\frac{1}{\alpha}+\frac{1}{\alpha}}{2\sqrt{\lambda_1}}\right)(g_p^\epsilon - g_p)\right]^2 \\ &\le 2C_1^2\alpha^{-2}\left(\|\varphi^\epsilon - \varphi\|^2 + \|g^\epsilon - g\|^2\right) \\ &\le 4C_1^2\alpha^{-2}\epsilon^2, \end{aligned}$$

where $C_1 = \max(1, \frac{1}{\sqrt{\lambda_1}})$.

Hence

$$\|W^\epsilon(.,y) - w^\epsilon(.,y)\| \le 2C_1\alpha^{-1}\epsilon.$$

Now, we return to the proof of Theorem 4. Since $0 \le t \le 1$, we have

$$\frac{1}{\alpha\sqrt{\lambda_p} + e^{-\sqrt{\lambda_p}t}} \le \frac{1}{\alpha\sqrt{\lambda_p} + e^{-\sqrt{\lambda_p}}}.$$

It is not difficult to prove the following inequality

$$\frac{1}{\alpha\sqrt{\lambda_p} + e^{-\sqrt{\lambda_p}}} \le \frac{1}{\alpha\ln(\frac{1}{\alpha})}, \quad \alpha \in (0,e). \tag{23}$$

Inequalites (9) and (20) follow

$$
\begin{aligned}
& u(x,t) - w^{\epsilon}(x,t) \\
& = \sum_{p=1}^{\infty} \Bigg[\left(\frac{e^{\sqrt{\lambda_p} t} - \frac{e^{\sqrt{\lambda_p} t}}{1+\alpha\sqrt{\lambda_p} e^{\sqrt{\lambda_p} t}}}{2} \right) \varphi_p + \left(\frac{e^{\sqrt{\lambda_p} t} - \frac{e^{\sqrt{\lambda_p} t}}{1+\alpha\sqrt{\lambda_p} e^{\sqrt{\lambda_p} t}}}{2\sqrt{\lambda_p}} \right) g_p \\
& \qquad + \int_0^t \frac{e^{\sqrt{\lambda_p}(t-s)} - \frac{e^{-\sqrt{\lambda_p} s}}{\alpha\sqrt{\lambda_p} + e^{-\sqrt{\lambda_p} t}}}{2\sqrt{\lambda_p}} f_p(s) ds \Bigg] X_p(x).
\end{aligned}
$$

$$
\begin{aligned}
= & \sum_{p=1}^{\infty} \left(\frac{e^{\sqrt{\lambda_p} t} - \frac{e^{\sqrt{\lambda_p} t}}{1+\alpha\sqrt{\lambda_p} e^{\sqrt{\lambda_p} t}}}{2} \right) \left[\varphi_p + \frac{g_p}{\sqrt{\lambda_p}} + \frac{1}{\sqrt{\lambda_p}} \int_0^t e^{-\sqrt{\lambda_p} s} f_p(s) ds \right] X_p(x) \\
= & \sum_{p=1}^{\infty} \left(\frac{\alpha\sqrt{\lambda_p} e^{\sqrt{\lambda_p} t}}{\alpha\sqrt{\lambda_p} + e^{-\sqrt{\lambda_p} t}} \right) \left[\varphi_p + \frac{g_p}{\sqrt{\lambda_p}} + \frac{1}{\sqrt{\lambda_p}} \int_0^t e^{-\sqrt{\lambda_p} s} f_p(s) ds \right] X_p(x). \qquad (24)
\end{aligned}
$$

Using (12), we get

$$
< u(x,t), X_p(x) > + \frac{1}{\sqrt{\lambda_p}} < u_t(x,t), X_p(x) > = e^{\sqrt{\lambda_p} t} \left(\varphi_p + \frac{g_p}{\sqrt{\lambda_p}} + \frac{1}{\sqrt{\lambda_p}} \int_0^t e^{-\sqrt{\lambda_p} s} f_p(s) ds \right). \qquad (25)
$$

It follows from (24) and (25) that

$$
u(x,t) - w^{\epsilon}(x,t) = \sum_{p=1}^{\infty} \frac{\alpha\sqrt{\lambda_p}}{\alpha\sqrt{\lambda_p} + e^{-\sqrt{\lambda_p} t}} \left(< u(x,t), X_p(x) > + \frac{1}{\sqrt{\lambda_p}} < u_t(x,t), X_p(x) > \right) X_p(x).
$$

Using inequality (23) yields

$$
| < u(x,t) - w^{\epsilon}(x,t), X_p(x) > | \le \frac{1}{\ln(\frac{1}{\alpha})} \left(\sqrt{\lambda_p} | < u(x,t), X_p(x) > | + | < u_t(x,t), X_p(x) > | \right).
$$

Using the inequalities $(a+b)^2 \le 2a^2 + 2b^2$, we obtain

$$
\begin{aligned}
\|u(.,t) - w^{\epsilon}(.,t)\|^2 &= \sum_{p=1}^{\infty} | < u(x,t) - w^{\epsilon}(x,t), X_p(x) > |^2 \\
&\le 2 \sum_{p=1}^{\infty} \frac{1}{\ln^2(\frac{1}{\alpha})} \left[\lambda_p | < u(x,t), X_p(x) > |^2 + | < u_t(x,t), X_p(x) > |^2 \right] \\
&\le \frac{2}{\ln^2(\frac{1}{\alpha})} \Big[\frac{1}{\lambda_1} \|Lu(.,t)\|^2 + \|u_t(.,t)\|^2 \Big].
\end{aligned}
$$

Applying the triangle inequality and Lemma 5, we obtain

$$
\begin{aligned}
\|u(.,t) - W^{\epsilon}(.,t)\| &\le \|u(.,t) - w^{\epsilon}(.,t)\| + \|w^{\epsilon}(.,t) - W^{\epsilon}(.,t)\| \\
&\le \frac{\sqrt{2} C_1}{\ln(\frac{1}{\alpha})} E + 2C_1 \alpha^{-1} \epsilon.
\end{aligned}
$$

Since $\alpha = \epsilon^a$, we obtain

$$\|u(.,t) - W^\epsilon(.,t)\| \leq \frac{\sqrt{2}C_1}{\ln(\frac{1}{\epsilon^a})}E + 2C_1\epsilon^{1-a}.$$

This completes the proof of Theorem 4. □

5. Numerical Results

As proved above, the proposed two regularization methods are completely stable. In this section, we give two examples to illustrate the behaviour of the proposed methods. Let us consider the following problem

$$\begin{cases} u_{tt} + u_{xx} = f(x,t), & (x,t) \in (0,\pi) \times (0,1), \\ u(0,t) = u(\pi,t) = 0, & t \in (0,1), \\ u_t(x,0) = g(x), & x \in (0,\pi), \\ u(x,0) = \varphi(x), & x \in (0,\pi). \end{cases} \tag{26}$$

In this case, the eigenvalues and eigenfunctions of Laplace operator are respectively given as follows

$$\lambda_p = p^2, \quad X_p(x) = \sqrt{\frac{2}{\pi}}\sin px.$$

Let $\epsilon > 0$, $\varphi^\epsilon, g^\epsilon$ are measured data such that

$$\left\|\varphi^\epsilon - \varphi\right\| \leq \epsilon, \qquad \left\|g^\epsilon - g\right\| \leq \epsilon.$$

Two proposed regularized solutions of Eq.(26) can be expressed as follows

$$u^{i,\epsilon}(x,t) = \sum_{p=1}^{\infty} u_p^{i,\epsilon}(t)\sin(px), \tag{27}$$

where

$$u_p^{i,\epsilon}(t) = E_p^{i,\epsilon}(t)\left(\frac{\varphi_p^\epsilon}{2} + \frac{g_p^\epsilon}{2p}\right) + e^{-tp}\left(\frac{\varphi_p^\epsilon}{2} - \frac{g_p^\epsilon}{2p}\right) + \frac{1}{\pi p}\int_0^t\int_0^\pi \left(F_p^{i,\epsilon}(t,s) - e^{p(s-t)}\right) f(\xi,s)\sin(p\xi)d\xi ds, \tag{28}$$

herein $\epsilon > 0$, and $i = 1, 2$ corresponding to ith method, i.e.,

- First method:

$$E_p^{1,\epsilon}(t) = e^{p(1-p\beta)t}, \qquad F_p^{1,\epsilon}(t,s) = e^{p(t-s-p\beta)}.$$

- Second method:

$$E_p^{2,\epsilon}(t) = \frac{1}{p\alpha + e^{-pt}}, \qquad F_p^{2,\epsilon}(t,s) = \frac{e^{-ps}}{p\alpha + e^{-pt}}.$$

The aim of this numerical work is to observe the following error

$$\delta^{i.\epsilon}(t) = \left\| u^{i,\epsilon}(\cdot, t) - u(\cdot, t) \right\|,$$

for $t \in [0, 1]$, as ϵ tends to zero. Applying Parseval's identity to the latter formulae, we have

$$\delta^{i.\epsilon}(t) = \sqrt{\frac{\pi}{2} \sum_{p=1}^{\infty} \left| u_p^{i,\epsilon}(t) - u_p(t) \right|^2}, \tag{29}$$

In order to estimate the right side of Eq.(28), without loss of generality, we consider a following general integral

$$I = \int_0^t \int_0^\pi G(x, s) \sin(px) dx ds, \tag{30}$$

where t, p are given numbers. The above integrand which including $\sin(px)$ is strongly oscillated as p increase (see e.g. [3]). Thus, we will here exploit an efficient method as mentioned in [2].

Put

$$h = \frac{\pi}{2n}, \;\; x_k = (k-1)h, \;\; k = 1, 2, ..., 2n+1, \tag{31}$$

where n is a given integer number, Eq.(30) becomes

$$I = \sum_{k=1}^{n} \int_0^t \int_{x_{2k-1}}^{x_{2k+1}} G(x, s) \sin(px) dx ds \tag{32}$$

We approximate G on $[x_{2k-1}, x_{2k+1}]$ by the quadratic interpolation and rewrite the oscillating term $\sin(px)$ as follows

$$G(x, s) \approx \left(\frac{r^2}{2} - \frac{r}{2} \right) G(x_{2k-1}, s) + (1 - r^2) G(x_{2k}, s) + \left(\frac{r^2}{2} + \frac{r}{2} \right) G(x_{2k+1}, s), \tag{33}$$

$$\sin(px) = \sin(px_{2k}) \cos(phr) + \cos(px_{2k}) \sin(phr), \tag{34}$$

where $x = x_{2k} + hr,\; r \in [-1, 1], s \in [0, t]$. Hence, Expressions (32),(33) and (34) yield

$$I \approx \sum_{k=1}^{n} \int_0^t \left[L_k G(x_{2k-1}, s) + M_k G(x_{2k}, s) + R_k G(x_{2k+1}, s) \right] ds \tag{35}$$

where the paramesters L_k, M_k, R_k are of the following forms

$$L_k = \left[\left(\frac{1}{\theta} - \frac{2}{\theta^2} \right) \sin\theta + \frac{2}{\theta^2} \cos\theta \right] a_k + \left[\frac{1}{\theta} \cos\theta - \frac{1}{\theta^2} \sin\theta \right] b_k,$$

$$M_k = 4 \left[\frac{1}{\theta^3} \sin\theta - \frac{1}{\theta^2} \cos\theta \right] a_k,$$

$$R_k = \left[\left(\frac{1}{\theta} - \frac{2}{\theta^2} \right) \sin\theta + \frac{2}{\theta^2} \cos\theta \right] a_k - \left[\frac{1}{\theta} \cos\theta - \frac{1}{\theta^2} \sin\theta \right] b_k,$$

with $\theta = ph$, $a_k = h\sin(px_{2k})$ and $b_k = h\cos(px_{2k})$. So far, the integrals in the right side of (35) was escaped from the oscillating factor $\sin(px)$, thus, will be estimated by Gauss-Legendre quadrature method [38].

In this paper, the program is written by Fortran90 [5] language with double-precision real type based on the IEEE Standard for floating-point number [6]. Due to the decay of Fourier series, the right side of Eq.(27) should be rounded by truncating cancelable terms in Fourier series as p becomes large, i.e., the series should be replaced by its Nth partial sum, where N is increased until the following condition is satisfied

$$\sum_{j=0}^{j_0-1} \left|u_{N-j}^{i,\epsilon}(t)\right| < \epsilon \max_{1\le k\le N} \left|u_k^{i,\epsilon}(t)\right|,$$

and $N > j_0$, where $t \in [0,1]$, $\epsilon > 0$ and j_0 is here chosen large enough ($\approx 10^2$). In order to avoid underflow error occuring to exponent function $e^{-\tau}$ as τ large (which is greater than 709), we replace unexpected result (maybe NaN, see [6]) by a tiny possitive number ($\approx 10^{-308}$); it also means that we eliminate invalid numbers in the right side of (28). In order to control the error of numerical integrating, we increase n in (31) and the number of abscissas of Gauss-Legendre method until getting desired accuracy.

Example 1.. Take $u(x,t) = x(\pi - x)e^{-tx}$ as the exact solution of the problem (26). Replacing u into (26) yields

$$f(x,t) = e^{-tx}\left(x^3(\pi - x) - t^2x^2 + 4tx + \pi t(tx - 2) - 2\right). \tag{36}$$

The measured datas are given

$$\varphi^\epsilon(x) = x(\pi - x) + \epsilon\sqrt{\frac{2}{\pi}}\sin x, \tag{37}$$

$$g^\epsilon(x) = x^2(\pi - x) + \epsilon\sqrt{\frac{2}{\pi}}\cos x, \tag{38}$$

where $\epsilon > 0$ denotes the error level, we can verify that

$$\left\|\varphi^\epsilon - \varphi\right\| = \left\|g^\epsilon - g\right\| = \epsilon.$$

In this example, we choose the regularization parameters of the 1st and 2nd method are $\beta = \frac{1}{\ln(1/\epsilon)}$ (i.e. $k = 1$) and $\alpha = \epsilon^{0.9}$ (i.e. $a = 0.9$), respectively.

The error estimation $\delta^{1,\epsilon}(t)$ and $\delta^{2,\epsilon}(t)$ at $\epsilon = 10^{-i}$ ($i = \overline{1,6}$), $t = (j-1)/10$ ($j = \overline{1,11}$) are provided in Table 1 and Table 2. By comparison, we observe that the convergence rate of the 2nd method is better than the first.

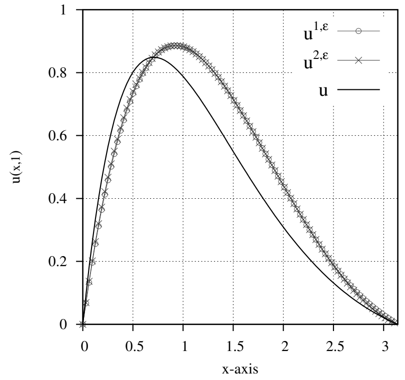
Morever, to illustrate more clearly the stability as well as the speed of convergence of the each method, Figure 1 which verifies what is presented by Tables 1 and 2 is given below. It includes two section cut, at $t = 1$ and 0.5, with $\epsilon = 10^{-1}, 10^{-2}$ and 10^{-3}.

t_j	$\epsilon = 10^{-1}$	$\epsilon = 10^{-2}$	$\epsilon = 10^{-3}$	$\epsilon = 10^{-4}$	$\epsilon = 10^{-5}$	$\epsilon = 10^{-6}$
0.0	1.00003E-01	1.00021E-02	1.00092E-03	1.00262E-04	1.00399E-05	1.01465E-06
0.1	1.19282E-01	2.59469E-02	1.45503E-02	1.07661E-02	8.62848E-03	7.20985E-03
0.2	1.35962E-01	4.09779E-02	2.59853E-02	1.96188E-02	1.58219E-02	1.32672E-02
0.3	1.48794E-01	5.33628E-02	3.56493E-02	2.72608E-02	2.21176E-02	1.86209E-02
0.4	1.58310E-01	6.36693E-02	4.40728E-02	3.40936E-02	2.78386E-02	2.35427E-02
0.5	1.65254E-01	7.23364E-02	5.15654E-02	4.03440E-02	3.31646E-02	2.81824E-02
0.6	1.70291E-01	7.96342E-02	5.82911E-02	4.61242E-02	3.81807E-02	3.26084E-02
0.7	1.73978E-01	8.57199E-02	6.43274E-02	5.14785E-02	4.29146E-02	3.68399E-02
0.8	1.76797E-01	9.06874E-02	6.97055E-02	5.64151E-02	4.73643E-02	4.08699E-02
0.9	1.79193E-01	9.45982E-02	7.44355E-02	6.09266E-02	5.15148E-02	4.46804E-02
1.0	1.81586E-01	9.74997E-02	7.85202E-02	6.50021E-02	5.53486E-02	4.82509E-02

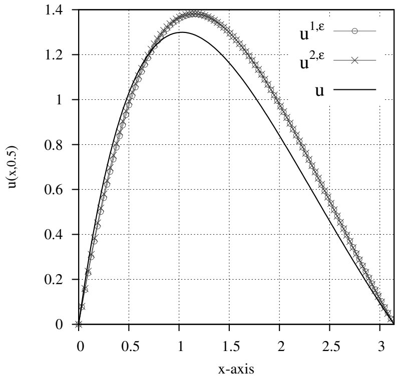
Table 1: Example 1, first method, error $\delta^{1,\epsilon}(t_j) = \left\| u^{1,\epsilon}(t_j) - u(t_j) \right\|$.

t_j	$\epsilon = 10^{-1}$	$\epsilon = 10^{-2}$	$\epsilon = 10^{-3}$	$\epsilon = 10^{-4}$	$\epsilon = 10^{-5}$	$\epsilon = 10^{-6}$
0.0	1.87132E-01	1.99644E-02	2.01084E-03	2.01312E-04	2.01513E-05	2.01928E-06
0.1	1.77903E-01	1.92356E-02	1.94246E-03	1.95625E-04	2.15202E-05	5.22828E-06
0.2	1.71001E-01	1.89103E-02	1.93955E-03	2.17208E-04	4.95520E-05	2.51458E-05
0.3	1.66404E-01	1.92172E-02	2.11939E-03	3.47587E-04	1.35076E-04	7.48558E-05
0.4	1.64034E-01	2.05085E-02	2.70238E-03	6.64758E-04	3.02585E-04	1.71522E-04
0.5	1.63676E-01	2.31238E-02	3.87542E-03	1.20832E-03	5.80238E-04	3.33328E-04
0.6	1.64973E-01	2.72078E-02	5.70916E-03	2.01935E-03	9.97673E-04	5.79174E-04
0.7	1.67493E-01	3.26513E-02	8.22071E-03	3.14136E-03	1.58466E-03	9.28425E-04
0.8	1.70816E-01	3.91713E-02	1.14078E-02	4.61428E-03	2.36983E-03	1.40049E-03
0.9	1.74634E-01	4.64178E-02	1.52443E-02	6.46810E-03	3.38002E-03	2.01435E-03
1.0	1.78771E-01	5.40370E-02	1.96846E-02	8.72410E-03	4.63658E-03	2.78686E-03

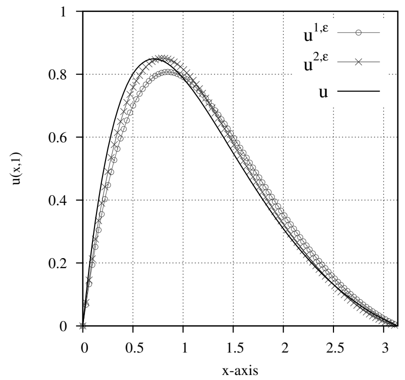
Table 2: Example 1, second method, error $\delta^{2,\epsilon}(t_j) = \left\| u^{2,\epsilon}(t_j) - u(t_j) \right\|$



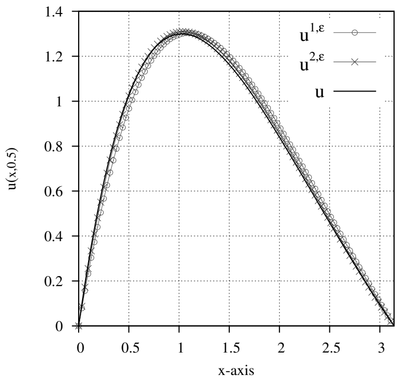
(a) $\epsilon = 10^{-1}$, at $t = 1$.



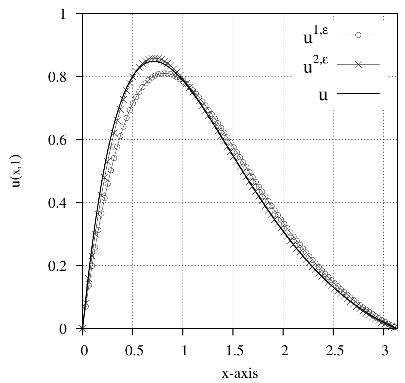
(b) $\epsilon = 10^{-1}$, at $t = 0.5$.



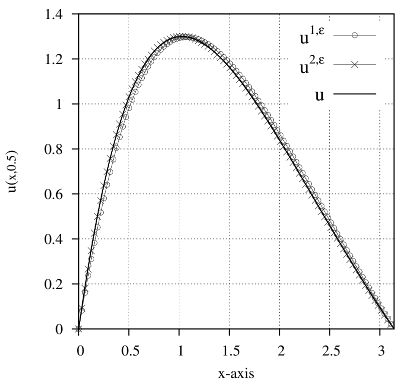
(c) $\epsilon = 10^{-2}$, at $t = 1$.



(d) $\epsilon = 10^{-2}$, at $t = 0.5$.



(e) $\epsilon = 10^{-3}$, at $t = 1$.



(f) $\epsilon = 10^{-3}$, at $t = 0.5$.

Figure 1: Example 1, graph of $u^{i,\epsilon}(\cdot, t)$ and $u(\cdot, t)$, at $t = 1$ and 0.5.

Example 2.. The same to previous example, we take $u(x,t) = \frac{x(\pi-x)}{x^2t+1}$ as the exact solution of the problem (26). Source function f and measured data are given as follows

$$f(x,t) = \frac{6tx^2 + 2\pi tx(tx^2-3) - 2 + 2(\pi - x)x^5}{(tx^2+1)^3}, \tag{39}$$

$$\varphi^\epsilon(x) = x(\pi - x) + \epsilon\sqrt{\frac{2}{\pi}}\sin 3x, \tag{40}$$

$$g^\epsilon(x) = x^3(\pi - x) + \epsilon\sqrt{\frac{2}{\pi}}\cos 4x. \tag{41}$$

Similarly as in Example 1, we here choose the regularization parameters of the 1st and 2nd method are $\beta = \frac{1}{\ln(1/\epsilon)}$ and $\alpha = \epsilon^{0.9}$, respectively. Table 3 and Table 4, respectively, show the error estimation $\delta^{1,\epsilon}(t)$ and $\delta^{2,\epsilon}(t)$ at $\epsilon = 10^{-i}$ $(i = \overline{1,6})$, $t = (j-1)/10$ $(j = \overline{1,11})$. Furthermore, Figure 2 is also given below to describe more clearly the graph of $u^{1,\epsilon}, u^{2,\epsilon}$ and u at two section cut $t = 1$ and 0.5. From these, we can state that the convergence of proposed methods are stable and the 2nd method is more effective than the first.

t_j	$\epsilon = 10^{-1}$	$\epsilon = 10^{-2}$	$\epsilon = 10^{-3}$	$\epsilon = 10^{-4}$	$\epsilon = 10^{-5}$	$\epsilon = 10^{-6}$
0.0	1.00003E-01	1.00021E-02	1.00092E-03	1.00264E-04	1.00638E-05	1.01465E-06
0.1	1.01314E-01	3.23198E-02	2.16587E-02	1.63924E-02	1.31566E-02	1.09833E-02
0.2	9.35319E-02	4.20283E-02	3.01196E-02	2.30705E-02	1.86014E-02	1.55709E-02
0.3	9.51640E-02	5.21332E-02	3.84276E-02	2.97034E-02	2.40706E-02	2.02171E-02
0.4	1.03748E-01	6.30133E-02	4.72873E-02	3.68670E-02	3.00327E-02	2.53153E-02
0.5	1.15296E-01	7.38959E-02	5.63595E-02	4.43236E-02	3.62997E-02	3.07100E-02
0.6	1.27396E-01	8.42162E-02	6.53165E-02	5.18302E-02	4.26763E-02	3.62385E-02
0.7	1.38951E-01	9.36420E-02	7.39345E-02	5.92161E-02	4.90229E-02	4.17833E-02
0.8	1.49585E-01	1.01990E-01	8.20686E-02	6.63685E-02	5.52453E-02	4.72632E-02
0.9	1.59288E-01	1.09168E-01	8.96244E-02	7.32140E-02	6.12801E-02	5.26225E-02
1.0	1.68240E-01	1.15147E-01	9.65380E-02	7.97052E-02	6.70853E-02	5.78228E-02

Table 3: Example 2, first method, error estimation $\delta^{1,\epsilon}(t_j) = \left\|u^\epsilon(t_j) - u(t_j)\right\|$

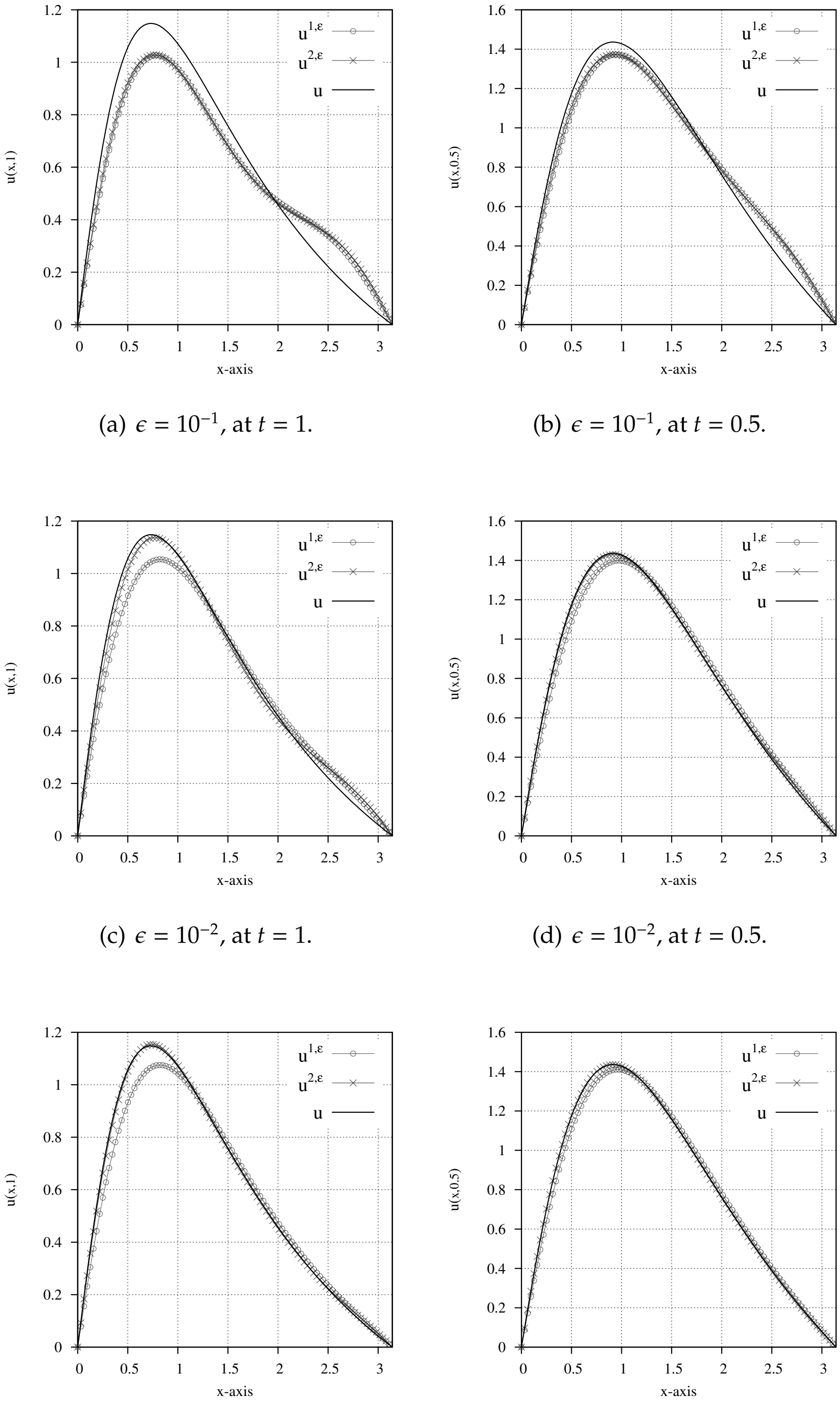
t_j	$\epsilon = 10^{-1}$	$\epsilon = 10^{-2}$	$\epsilon = 10^{-3}$	$\epsilon = 10^{-4}$	$\epsilon = 10^{-5}$	$\epsilon = 10^{-6}$
0.0	3.55477E-01	3.98561E-02	4.03659E-03	4.04219E-04	4.04364E-05	4.04576E-06
0.1	1.75053E-01	2.00949E-02	2.04219E-03	2.04967E-04	2.11374E-05	4.30370E-06
0.2	1.21079E-01	1.46422E-02	1.50833E-03	1.60976E-04	3.01266E-05	1.46989E-05
0.3	1.04816E-01	1.37371E-02	1.47318E-03	1.97685E-04	6.61859E-05	3.66514E-05
0.4	1.04438E-01	1.51570E-02	1.74455E-03	2.99857E-04	1.23390E-04	7.02401E-05
0.5	1.11195E-01	1.82061E-02	2.30269E-03	4.59512E-04	2.02156E-04	1.16424E-04
0.6	1.21127E-01	2.27009E-02	3.23605E-03	6.87384E-04	3.05028E-04	1.76408E-04
0.7	1.32370E-01	2.85469E-02	4.71239E-03	1.02156E-03	4.37168E-04	2.51765E-04
0.8	1.43988E-01	3.55433E-02	6.91672E-03	1.54142E-03	6.11036E-04	3.44992E-04
0.9	1.55465E-01	4.33455E-02	9.97094E-03	2.36925E-03	8.55374E-04	4.61211E-04
1.0	1.66484E-01	5.15379E-02	1.38961E-02	3.63190E-03	1.22679E-03	6.13544E-04

Table 4: Example 2, second method, error estimation $\delta^{2,\epsilon}(t_j) = \left\|u^\epsilon(t_j) - u(t_j)\right\|$

6. Conclusion

We have considered two regularization problems for a Cauchy problem for elliptic equation with inhomogeneous source, namely Problem (1). We also establish the error estimate of logarithmic type for all $t \in [0, T]$ under assumptions of the exact solution. This work improves many earlier results. In the future work, we will consider a generalized problem as follow

$$\begin{cases} u_{tt} = Lu + f(x, t, u(x, t)), (x, t) \in \Omega \times (0, 1), \\ u(x, t) = 0, t \in (0, 1), x \in \partial\Omega, \\ u(x, 0) = \varphi(x), x \in \Omega \\ u_t(x, 0) = g(x), x \in \Omega. \end{cases}$$



(a) $\epsilon = 10^{-1}$, at $t = 1$.

(b) $\epsilon = 10^{-1}$, at $t = 0.5$.

(c) $\epsilon = 10^{-2}$, at $t = 1$.

(d) $\epsilon = 10^{-2}$, at $t = 0.5$.

(e) $\epsilon = 10^{-3}$, at $t = 1$.

(f) $\epsilon = 10^{-3}$, at $t = 0.5$.

Figure 2: Example 2, graph of $u^{i,\epsilon}(\cdot, t)$ and $u(\cdot, t)$, at $t = 1$ and 0.5.

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