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Further Investigations on Fujimoto Type Strong Uniqueness Polynomials

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Abstract. Taking the question posed by the first author in [1] into background, we further exhaust-ably investigate existing Fujimoto type Strong Uniqueness Polynomial for Meromorphic functions (SUPM). We also introduce a new kind of SUPM named Restricted SUPM and exhibit some results which will give us a new direction to discuss the characteristics of a SUPM. Moreover, throughout the paper, we pose a number of open questions for future research.

1. Introduction Definitions and Results

Let us denote by $\mathbb{C}$, the set of all complex numbers and $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Also by any meromorphic function f, we always mean that it is defined on $\mathbb{C}$. Here we consider the standard notations of Nevanlinna theory as explained in [7]. For any non-constant meromorphic function $h(z)$, we define $S(r,h)$ by $S(r,h) = o(T(r,h))$ *as* $r \longrightarrow \infty$, $r \notin E$, where E denotes any set of positive real numbers having finite linear measure.

For any two non-constant meromorphic functions f, g and $a \in \mathbb{C}$, we say that f and g share the value a-CM (counting multiplicities) if the zeros of $f-a$ and $g-a$ coincides in location as well as in multiplicities. Also we say that f and g share the value a-IM (ignoring multiplicities), if the zeros of $f-a$ and $g-a$ coincide in location only.

In addition, we say that f and g share ∞-CM (resp. IM), if $1/f$ and $1/g$ share 0-CM (resp. IM).

About ninety years ago, R. Nevanlinna, the founder of value distribution theory, proved his famous Five Value and Four Value theorems which were the inception of uniqueness theory. After fifty years or so generalizing the value sharing problems to the set sharing problems which focused mainly on the study of uniqueness of two entire or meromorphic functions via pre-image sets, F. Gross started a new era of uniqueness theory. Though this paper is devoted to the transition of set sharing problems but initially we shortly recall the following two standard definitions from the literature.

Definition 1.1. *Let $S \subseteq \mathbb{C} \cup \{\infty\}$ and $E_f(S) = \bigcup_{a \in S}\{z : f(z) = a\}$, where each zero of $f - a$ is counted according to its multiplicity. We say that f and g share the set S-CM if $E_f(S) = E_g(S)$.*

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Evidently, if S contains only one element, then it coincides with the usual definition of CM sharing of values.

Definition 1.2. *Let $S \subset \mathbb{C} \cup \{\infty\}$; f and g be two non-constant meromorphic (resp. entire) functions. If $E_f(S) = E_g(S)$ implies $f \equiv g$, then S is called a unique range set for meromorphic (resp. entire) functions or in brief URSM (resp. URSE).*

Apropos of the set sharing problems, in 1982, F. Gross and C. C. Yang [6] first ensured the existence of a unique range set which is as follows:

Theorem A. *[6] Let $S = \{z \in \mathbb{C} : e^z + z = 0\}$. If two entire functions f and g satisfy $E_f(S) = E_g(S)$, then $f \equiv g$.*

It is to be observed that the range set S given in *Theorem A* is an infinite set. So later on a lot of investigations were made by Li-Yang, Yi, Frank-Reinders and the First Author in [8], [10], [3] and [1] respectively to find finite unique range sets with smallest cardinality.

In relation to this, Li-Yang [8] first exhaust-ably delve into the set sharing problems and retrieve the matter to a completely different scenario that finite URSM's are nothing but the set of distinct zeros of some suitable polynomials; i.e., if $S = \{a_1, a_2, \ldots, a_n\}$, then it is necessary for S to be a unique range set for meromorphic (resp. entire) functions that the associated polynomial $P_S(w) = (w - a_1)(w - a_2)\ldots(w - a_n)$ satisfies the condition $P(f) = P(g)$. In course of time, the set sharing problems have gradually been transformed to a new direction and eventually it turns towards characterization of polynomials generating URSM (resp. URSE).

Under this new perspective, we first invoke the following two definitions:

Definition 1.3. *A polynomial $P(z)$ in $\mathbb{C}$, is called a uniqueness polynomial for meromorphic (resp. entire) functions, if for any two non-constant meromorphic (resp. entire) functions f and g, $P(f) \equiv P(g)$ implies $f \equiv g$. We say $P(z)$ is a UPM (resp. UPE) in brief.*

Definition 1.4. *A polynomial $P(z)$ in $\mathbb{C}$ is called a strong uniqueness polynomial for meromorphic (resp. entire) functions if for any two non-constant meromorphic (resp. entire) functions f and g, $P(f) \equiv \kappa P(g)$ implies $f \equiv g$, where κ is any non-zero constant. In this case, we say $P(z)$ is a SUPM (resp. SUPE) in brief.*

In 2000, to find the necessary and sufficient conditions for a monic polynomial having only simple zeros to be a UPM, Fujimoto [4] made a major breakthrough by introducing a new idea namely "Property H" which has recently been justified as "Critical injection property" in [2]. The definition is as follows:

Definition 1.5. *Let $P(z)$ be a polynomial such that $P'(z)$ has mutually k distinct zeros given by $d_1, d_2, \ldots, d_k$ with multiplicities $q_1, q_2, \ldots, q_k$ respectively. Then $P(z)$ is said to satisfy critical injection property if $P(d_i) \neq P(d_j)$ for $i \neq j$, where $i, j \in \{1, 2, \ldots, k\}$.*

From the definition, it is obvious that $P(z)$ is injective on the set of distinct zeros of $P'(z)$ which are known as critical points of $P(z)$. Furthermore, any polynomial $P(z)$ satisfying this property is called as "critically injective polynomial". Thus a critically injective polynomial has at-most one multiple zero.

Using this fundamental property, in [5], Fujimoto completely characterized monic polynomials with only simple zeros to be a uniqueness polynomial.

Theorem B. *[5] Suppose that $P(z)$ is critically injective. Then $P(z)$ will be a uniqueness polynomial if and only if*

$$\sum_{1 \leq l < m \leq k} q_l q_m > \sum_{l=1}^{k} q_l.$$

In particular, the above inequality is always satisfied whenever $k \geq 4$. When $k = 3$ and $\max\{q_1, q_2, q_3\} \geq 2$ or when $k = 2$, $\min\{q_1, q_2\} \geq 2$ and $q_1 + q_2 \geq 5$, then also the above inequality holds.

Moreover, in [4], Fujimoto proved that critical injection property suffices the set of zeros S of a SUPM (resp. SUPE) to be a URSM (resp. URSE).

So it is needless to say that UPM (resp. UPE) and SUPM (resp. SUPE) both play a pivotal role in finding unique range sets. In this context, recently the first author [1] introduced a strong uniqueness polynomial whose zero set is also forming an unique range set. In the same paper, the first author posed a question:

Question A. *"Does there exist any critically injective* SUPM *with degree less than* 7 *?"*

Motivated by the above question, the current paper has been organized. So at first we recall the polynomial generating URSM introduced by Frank and Reinders in [3].

$$P_{FR}(z) = \frac{(n-1)(n-2)}{2}z^n - n(n-2)z^{n-1} + \frac{n(n-1)}{2}z^{n-2} - c \;\; (c \neq 0, 1).$$

Obviously, $P_{FR}(z)$ is critically injective. Also in their paper, Frank-Reinders proved that $P_{FR}(z)$ is a SUPM for $n \geq 8$, when $c \neq 0, 1$; i.e., when $P_{FR}(z)$ has only simple zero. But this does not commensurate with the actual definition of SUPM where restrictions over multiplicity are not taken into account. So natural question arises what would happen if we consider multiple zeros of $P_{FR}(z)$? In this regard, we have the following theorem which is a direct improvement of the above result.

Theorem 1.1. *Let* $P_{FR_1}(z) = \frac{(n-1)(n-2)}{2}z^n - n(n-2)z^{n-1} + \frac{n(n-1)}{2}z^{n-2} - c$*, where* $c \in \mathbb{C}$*. Then* $P_{FR_1}(z)$ *is a critically injective SUPM for* $n \geq 8$*.*

The following *Lemmas* are needed in this sequel.

Lemma 1.1. *[9] Let* f *be a non-constant meromorphic function and let*

$$R(f) = \frac{\sum\limits_{k=0}^{n} a_k f^k}{\sum\limits_{j=0}^{m} b_j f^j}$$

be an irreducible rational function in f *with constant coefficients* $\{a_k\}$ *and* $\{b_j\}$*, where* $a_n \neq 0$ *and* $b_m \neq 0$*. Then*

$$T(r, R(f)) = dT(r, f) + S(r, f),$$

where $d = \max\{n, m\}$*.*

Lemma 1.2. *If*

$$\psi(t) = (n-1)^2(t^n - \kappa)(t^{n-2} - \kappa) - n(n-2)(t^{n-1} - \kappa)^2$$

and $\kappa \neq 0, 1$*; then* $\psi(t) = 0$ *has no multiple root.*

Proof. **[Proof]** Let $F(t) = \psi(e^t)e^{(1-n)t}$ for $t \in \mathbb{C}$. Then by elementary calculations, we get

$$F(t) = (e^{(n-1)t} + \kappa^2 e^{-(n-1)t}) - \kappa(n-1)^2(e^t + e^{-t}) + 2\kappa n(n-2).$$

Thus $\psi(t) \neq 0$ for $t = 0$. Next, if possible, let $\psi(z_0) = \psi'(z_0) = 0$ for some $z_0 \in \mathbb{C}$.
Then $z_0 \neq 0$, and hence there exist some $w_0 \in \mathbb{C}$ such that $z_0 = e^{w_0}$.
As $F'(t) = \psi'(e^t)e^{(1-n)t}e^t - (n-1)\psi(e^t)e^{(1-n)t}$, so $F(w_0) = F'(w_0) = 0$.
Thus

$$(e^{(n-1)w_0} + \kappa^2 e^{-(n-1)w_0}) = \kappa(n-1)^2(e^{w_0} + e^{-w_0}) - 2\kappa n(n-2),$$

and

$$(e^{(n-1)w_0} - \kappa^2 e^{-(n-1)w_0}) = \kappa(n-1)(e^{w_0} - e^{-w_0}).$$

Therefore

$$\begin{aligned} 4\kappa^2 &= (e^{(n-1)w_0} + \kappa^2 e^{-(n-1)w_0})^2 - (e^{(n-1)w_0} - \kappa^2 e^{-(n-1)w_0})^2 \\ &= (\kappa(n-1)^2(e^{w_0} + e^{-w_0}) - 2\kappa n(n-2))^2 - (\kappa(n-1)(e^{w_0} - e^{-w_0}))^2 \\ &= 4\kappa^2\{\left((n-1)^2\cosh w_0 - n(n-2)\right)^2 - ((n-1)\sinh w_0)^2\}, \end{aligned}$$

i.e.,

$$(\cosh w_0)^2\{(n-1)^4 - (n-1)^2\} = 2n(n-1)^2(n-2)\cosh w_0 - \{n^2(n-2)^2 + (n-1)^2 - 1\},$$

i.e., $(\cosh w_0 - 1)^2 = 0$, that is, $\cosh w_0 = 1$, which implies $z_0 + \frac{1}{z_0} = 2$.
Hence $z_0 = 1$ but $\psi(1) = (1-\kappa)^2 \neq 0$ as $\kappa \neq 1$. Thus our assumption is wrong. Hence the proof. □

Lemma 1.3. *If*

$$\psi(t) = (n-1)^2(t^n - \kappa)(t^{n-2} - \kappa) - n(n-2)(t^{n-1} - \kappa)^2,$$

where $t \neq 1$ and $\kappa \neq 0, 1$; then $\psi(t) = 0$ and $t^n - \kappa = 0$ has no common root.

Proof. **[Proof]** If $\psi(t) = 0$ and $t^n - \kappa = 0$ has a common root, then by the expression of $\psi(t)$, we get $t^{n-1} - \kappa = 0$ and $t^n - \kappa = 0$. Then $\kappa = t^n = tt^{n-1} = t\kappa$, which is impossible as $\kappa \neq 0$ and $t \neq 1$. Hence the proof. □

Proof. **[Proof of Theorem 1.1]** It keeps nothing to prove that $P_{FR_1}(z)$ is critically injective. Since

$$P'_{FR_1}(z) = \frac{n(n-1)(n-2)}{2} z^{n-3}(z-1)^2$$

so obviously

$$P_{FR_1}(z) + c - 1 = (z-1)^3 \prod_{i=1}^{n-3}(z - \eta_i),$$

where $\eta_i \neq 1$ for all $i = 1, 2, \ldots, n-3$ and

$$P_{FR_1}(z) + c = z^{n-2}(z - \alpha_1)(z - \alpha_2),$$

where $\alpha_i \neq 0$ for $i = 1, 2$. Suppose f and g be two non-constant meromorphic functions such that $P_{FR_1}(g) = \kappa P_{FR_1}(f)$, where $\kappa \in \mathbb{C} \setminus \{0\}$. Using *Lemma 1.1*, we get

$$T(r, f) = T(r, g) + O(1). \tag{1.1}$$

Now we have the following cases:

Case-1 Let $\kappa \neq 1$.

Subcase-1.1. If $c \neq 0$, then we consider the following subcases:

Subcase-1.1.1. Let $c \neq 1$. Then proceeding same as in the line of proof of [p. 191, Case-2, [3]], we get contradiction for $n \geq 8$.

Subcase-1.1.2. Let $c = 1$. Then we have $P_{FR_1}(f) + \frac{1}{\kappa} = \frac{1}{\kappa} g^{n-2}(g - \alpha_1)(g - \alpha_2)$. Clearly $\frac{1}{\kappa} \neq 1$ i.e., $\frac{1}{\kappa} \neq c$ and $\frac{1}{\kappa} \neq 0$, i.e., $\frac{1}{\kappa} \neq c - 1$. So $P_{FR_1}(f) + \frac{1}{\kappa}$ has only simple zeros, say δ_i for $i = 1, 2, \ldots, n$. Therefore we have

$$\prod_{i=1}^{n}(f - \delta_i) = \frac{1}{\kappa} g^{n-2}(g - \alpha_1)(g - \alpha_2).$$

Now using the second fundamental theorem and (1.1), we get

$$\begin{aligned} (n-2)T(r, f) &\leq \sum_{i=1}^{n} \overline{N}(r, \delta_i; f) + S(r, f) \qquad (1.2) \\ &\leq \overline{N}(r, 0; g) + \overline{N}(r, \alpha_1; g) + \overline{N}(r, \alpha_2; g) + S(r, f), \end{aligned}$$

which is a contradiction for $n \geq 6$.

Subcase-1.2. If $c = 0$, then we have

$$\frac{(n-1)(n-2)}{2}(f^n - \kappa g^n) - n(n-2)(f^{n-1} - \kappa g^{n-1}) + \frac{n(n-1)}{2}(f^{n-2} - \kappa g^{n-2}) = 0.$$

By putting $h = \frac{f}{g}$, we get

$$\frac{(n-1)(n-2)}{2}(h^n - \kappa)g^2 - n(n-2)(h^{n-1} - \kappa)g + \frac{n(n-1)}{2}(h^{n-2} - \kappa) = 0. \tag{1.3}$$

If h is constant, then as g is non-constant, so from (1.3), we get $h^n - \kappa = 0$, $h^{n-1} - \kappa = 0$ and $h^{n-2} - \kappa = 0$. So $\kappa = h^n = h.h^{n-1} = h\kappa$, i.e., $h = 1$ and hence $f \equiv g$. If h is non-constant, then again from equation (1.3), we have

$$\left(g - \frac{n}{n-1}\frac{h^{n-1} - \kappa}{h^n - \kappa}\right)^2 = -\frac{n\psi(h)}{(n-1)^2(n-2)(h^n - \kappa)^2}, \tag{1.4}$$

where $\psi(h)$ is a polynomial of degree $2n - 2$ defined by

$$\psi(t) = (n-1)^2(t^n - \kappa)(t^{n-2} - \kappa) - n(n-2)(t^{n-1} - \kappa)^2.$$

By using *Lemmas (1.2) and (1.3)*, we can write equation (1.4) as

$$\left(g - \frac{n}{n-1}\frac{h^{n-1} - \kappa}{h^n - \kappa}\right)^2 = -n\frac{\prod_{i=1}^{2n-2}(h - \alpha_i)}{(n-1)^2(n-2)(h^n - \kappa)^2}, \tag{1.5}$$

where α_i's are mutually distinct zeros of $\psi(h)$.

Thus using the second fundamental theorem, we get

$$\begin{aligned}(2n-4)T(r,h) &\leq \sum_{i=1}^{2n-2}\overline{N}(r,\alpha_i;h) + S(r,h)\\ &\leq \frac{1}{2}\sum_{i=1}^{2n-2}N(r,\alpha_i;h) + S(r,h)\\ &\leq (n-1)T(r,h) + S(r,h),\end{aligned}$$

which is a contradiction for $n \geq 4$.

Case-2 Let $\kappa = 1$. This case also can be resorted same as in the line of proof of (p. 191, Case-3, [3]) and we can get $f \equiv g$ for $n \geq 6$.

Hence the proof. □

Remark 1.1. *Applying* part-b of Theorem-8 of [8] *and* Subcase-1.2. of the Theorem 1.1, *it is easy to verify that* $\frac{(n-1)(n-2)}{2}z^n - n(n-2)z^{n-1} + \frac{n(n-1)}{2}z^{n-2}$ *is a four degree SUPE.*

Next we consider the following polynomial introduced by the first author in [1].

$$P_B(z) = \sum_{i=0}^{m}\binom{m}{i}\frac{(-1)^i}{n+m+1-i}z^{n+m+1-i} + c = Q(z) + c,$$

where $Q(z) = \sum_{i=0}^{m}\binom{m}{i}\frac{(-1)^i}{n+m+1-i}z^{n+m+1-i}$ and c $(\neq 0, -Q(1))$ is a constant.

In [1], the first author proved that $P_B(z)$ is a critically injective SUPM when $n \geq 3$, $m \geq 3$ and $c = 1$, i.e., a SUPM of degree 7. But had the *Theorem 1.1* in [1] been proved independently for the case $c = 1$, it could have been proved that $P_B(z)$ is a SUPM for $n = 3$, $m = 2$ and $c = 1$ i.e., a SUPM of degree 6.

One more thing is to be noticed that if $m = 2$, then $P_B(z)$ reduces to

$$P_B(z) = \frac{z^{n+3}}{n+3} - 2\frac{z^{n+2}}{n+2} + \frac{z^{n+1}}{n+1} + c,$$

where $c \neq 0, -\frac{2}{(n+1)(n+2)(n+3)}$.

Multiplying by $\frac{(n+1)(n+2)(n+3)}{2}$ to the above polynomial, we get

$$P_{B_1}(z) = \frac{(n+1)(n+2)}{2}z^{n+3} - (n+1)(n+3)z^{n+2} + \frac{(n+2)(n+3)}{2}z^{n+1} - c_1,$$

where $c_1 = -c\frac{(n+1)(n+2)(n+3)}{2}$. Now putting $n + 3 = n_1$, we have

$$P_{B_1}(z) = \frac{(n_1-1)(n_1-2)}{2}z^{n_1} - n_1(n_1-2)z^{n_1-1} + \frac{n_1(n_1-1)}{2}z^{n_1-2} - c_1,$$

where $c_1 = -c\frac{n_1(n_1-1)(n_1-2)}{2}$ and $c \neq 0, -\frac{2}{n_1(n_1-1)(n_1-2)}$, i.e., $c_1 \neq 0, 1$, which is nothing but $P_{FR}(z)$. So $P_B(z)$ is a generalization of $P_{FR}(z)$. In [1], first author could prove that $P_{FR}(z)$ is a critically injective SUPM of degree 6, only for $c_1 = -60$. But in the next theorem, we shall prove it in more general settings.

Theorem 1.2. $P_{FR_2}(z) = \frac{(n-1)(n-2)}{2}z^n - n(n-2)z^{n-1} + \frac{n(n-1)}{2}z^{n-2} - c$, *where* $c \in \mathbb{C} \setminus \{\frac{1}{2}\}$ *is a critically injective SUPM for* $n \geq 6$.

Proof. **[Proof]** Let f and g be two non-constant meromorphic functions such that $P_{FR_2}(f) = \kappa P_{FR_2}(g)$ for $\kappa \in \mathbb{C} \setminus \{0\}$. Using *Lemma 1.1*, we easily get (1.1). Let

$$P_1(z) = \frac{(n-1)(n-2)}{2}z^n - n(n-2)z^{n-1} + \frac{n(n-1)}{2}z^{n-2}.$$

Therefore $P_{FR_2}(f) = \kappa P_{FR_2}(g)$ implies

$$P_1(f) - c = \kappa(P_1(g) - c). \tag{1.6}$$

Now we consider the following cases:

Case-1 Let $\kappa \neq 1$. Then we discuss the following subcases:

Subcase-1.1. If $c = 0$, then we can resolve it similarly as done in Subcase 1.2. of *Theorem 1.1* and get a contradiction for $n \geq 6$.

Subcase-1.2. If $c \neq 0$, then from (1.6), we get

$$P_1(f) = \kappa\left(P_1(g) - \frac{c(\kappa-1)}{\kappa}\right).$$

Clearly $\frac{c(\kappa-1)}{\kappa} \neq 0$ as $c \neq 0$. Now we have the following subcases:

Subcase-1.2.1. Let $\frac{c(\kappa-1)}{\kappa} \neq 1$. So $P_1(g) - \frac{c(\kappa-1)}{\kappa}$ has only simple zeros and let us denote them by δ_i' for $i = 1, 2, \ldots, n$. That is, we get

$$f^{n-2}(f-\alpha_1)(f-\alpha_2) = \kappa\prod_{i=1}^{n}(g-\delta_i').$$

Therefore by the second fundamental theorem and (1.1), we get

$$\begin{aligned}(n-2)T(r,g) &\leq \sum_{i=1}^{n}\overline{N}(r,\delta_i';g) + S(r,g) \\ &\leq \overline{N}(r,0;f) + \overline{N}(r,\alpha_1;f) + \overline{N}(r,\alpha_2;f) + S(r,g),\end{aligned} \tag{1.7}$$

which is a contradiction for $n \geq 6$.

Subcase-1.2.2. Let $\frac{c(\kappa-1)}{\kappa} = 1$. Then from (1.6), we get

$$P_1(f) - c(1-\kappa) = \kappa P_1(g).$$

Obviously $c(1-\kappa) \neq 0$. Now we claim that $c(1-\kappa) \neq 1$ because if $c(1-\kappa) = 1$, then $\kappa = \frac{c-1}{c}$. We also have $\frac{c(\kappa-1)}{\kappa} = 1$, i.e., $\kappa = \frac{c}{c-1}$. Therefore $\frac{c-1}{c} = \frac{c}{c-1}$, i.e., $c = \frac{1}{2}$, a contradiction. Hence $P_1(f) - c(1-\kappa) = \kappa P_1(g)$ has only simple zeros and let them be δ'', for $i = 1, \ldots, n$, i.e., we have

$$\prod_{i=1}^{n} (f - \delta_i'') = \kappa g^{n-2}(g - \alpha_1)(g - \alpha_2).$$

Again using the second fundamental theorem and (1.1), we have

$$\begin{aligned} (n-2)T(r,f) &\leq \sum_{i=1}^{n} \overline{N}(r, \delta''; f) + S(r,f) \\ &\leq \overline{N}(r,0;g) + \overline{N}(r,\alpha_1;g) + \overline{N}(r,\alpha_2;g) + S(r,f), \end{aligned} \tag{1.8}$$

which is a contradiction for $n \geq 6$.

Case-2 Let $\kappa = 1$. Then proceeding same as resorted in (p. 191, Case-3, [3]), we can get $f \equiv g$ for $n \geq 6$. Thus the proof follows. □

The above theorem clearly implies that if we assume $c \in \mathbb{C} \setminus \frac{1}{2}$ in $P_{FR}(z)$, then degree of the SUPM can be reduced significantly. So natural question arises:

Question 1.1. *Is it possible to reduce the degree of SUPM*

$$P_{FR}(z) = \frac{(n-1)(n-2)}{2} z^n - n(n-2)z^{n-1} + \frac{n(n-1)}{2} z^{n-2} - \frac{1}{2}$$

upto 6 ?

Now let us define the following notion:

Definition 1.6. *Let*

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

be a polynomial with the property that $a_n \neq 0$. Then $p(z)$ is called an initial term gap polynomial (ITGP) if $a_i = 0$ but $a_j \neq 0$ for at least one j such that $1 \leq j < i < n$; and for an initial term non-gap polynomial (ITNGP), there does not exist any such i.

So certainly $P_B(z)$ and $P_{FR}(z)$ are examples of ITNGP and $P_Y(z) = z^n + az^{n-r} + b$, the polynomial demonstrated by Yi [10] is an example of ITGP. Till now we have discussed and improved previous results of SUPM for existing ITNGP so our next discussion naturally turns towards SUPM for existing ITGP.

Henceforth, let us recapitulate $P_Y(z) = z^n + az^{n-r} + b$, where n, r are two positive integers having no common factors, $r \geq 2$ and $a(\neq 0)$, $b(\neq 0)$ are so chosen that $P_Y(z)$ has n distinct zeros. Yi proved that $P_Y(z)$ is a SUPM for $n \geq 2r + 4$ (see p.79, Case 3, first part, [10]).

Moreover, in 2000, Fujimoto showed that $P_Y(z)$ is a SUPM for $n > r + 1$, when $r \geq 3$ (see p. 1192, example 4.10., [4]). For $r = 2$, we prove the next theorem.

Remark 1.2. *The question posed by the first author in [1] would have been more relevant for ITNGP.*

Theorem 1.3. *If $P(z) = z^5 + az^3 + b$, where $ab \neq 0$ and a, b are chosen in such a way that $P(z)$ has only simple zero, then $P(z)$ is a critically injective SUPM if $\frac{a^5}{b^2} \neq -\frac{5^5}{3^3}$.*

Remark 1.3. *Though Theorem 1.3 can be proved as an application of Theorem 3.2, case-3 (p.42, [5]) yet for the sake of convenience we give the proof in detail.*

We need the following *Lemma* to proceed further.

Lemma 1.4. *[5](P. 42,* Lemma *3.3) Let $P(z)$ be a critically injective monic polynomial such that*

$$P'(z) = A(z-d_1)^{q_1}(z-d_2)^{q_2}...(z-d_k)^{q_k},$$

where $d_1, d_2, \ldots, d_k$ are mutually distinct, $A \in \mathbb{C} \setminus \{0\}$ and $P(z)$ has only simple zeros with $k = 3$ satisfying the condition $P(f) = cP(g)$ for any two non-constant meromorphic functions f and g, where c is any non-zero constant. If for $c \neq 1$, $\Lambda := \{(l, m) \,|P(d_l) = cP(d_m)\}$, then there are some indices m and m' such that $(l_0, m) \in \Lambda$ and $(m', l_0) \in \Lambda$ for some l_0 satisfying $q_{l_0} \geq 2$.

Proof. **[Proof of Theorem 1.3]** Undoubtedly $P(z)$ is a special form of $P_Y(z)$ for $n = 5$ and $r = 2$, so it is critically injective. Let f and g be two non-constant meromorphic functions such that $P(f) \equiv \kappa P(g)$, where $\kappa \in \mathbb{C} \setminus \{0\}$. Now we consider two cases:

Case-1 Let $\kappa \neq 1$. In this case, $P'(z) = 5z^2(z^2 + \frac{3a}{5})$. Let d_1 and d_2 are the zeros of the equation $(z^2 + \frac{3a}{5}) = 0$ and $d_3 = 0$. Clearly, $P(d_3) = b$ and $d_1 + d_2 = 0$.

Now, by the *Lemma 1.4*, there are indices m, m' such that

$$P(d_3) = \kappa P(d_m) \;\; and \;\; P(d_{m'}) = \kappa P(d_3), \tag{1.9}$$

where $\{m, m'\} \in \{1, 2, 3\}$.

If **m=3**, then by the equation (1.9), we get $\kappa = 1$, which is a contradiction.

Similarly, if $m' = 3$, then $\kappa = 1$, a contradiction.

If **m=2**, then either $m' = 2$ or $m' = 1$.

In this case, if $m' = 1$, then by the equation (1.9), we get $P(d_2) = \frac{b}{\kappa}$ and $P(d_1) = b\kappa$.

Now

$$\begin{aligned} P(d_2) &= P(-d_1) \\ &= -d_1^5 - ad_1^3 + b \\ &= -d_1^5 - ad_1^3 - b + 2b. \end{aligned}$$

Thus

$$P(d_2) + P(d_1) = 2b.$$

As $b \neq 0$ and $\kappa \neq 0$, we get $\kappa^2 - 2\kappa + 1 = 0$, i.e., $\kappa = 1$, a contradiction.

If $m' = 2$, then by the equation (1.9), we get $P(d_2) = \frac{b}{\kappa}$ and $P(d_2) = b\kappa$, which gives $\kappa^2 = 1$, i.e., $\kappa = \pm 1$.

If $\kappa = -1$, then $P(d_2) = -b$, which implies $d_2^3(d_2^2 + a) = -2b$. So squaring on both sides and then putting the value of d_2^2, we get $\frac{a^5}{b^2} = -\frac{5^5}{3^3}$, a contradiction.

If **m=1**, then either $m' = 2$ or $m' = 1$. Then proceeding similarly as above case, we arrive at contradiction.

Case-2 Let $\kappa = 1$. Then we have

$$f^5 + af^3 \equiv g^5 + ag^3. \tag{1.10}$$

By putting $f = gh$ in above, we get

$$g^2(h^5 - 1) \equiv -a(h^3 - 1). \tag{1.11}$$

First we assume that h is a non-constant function. Then we can write (1.11) as

$$g^2 \equiv -a\frac{(h-v)(h-v^2)}{(h-u)(h-u^2)(h-u^3)(h-u^4)}, \tag{1.12}$$

where $u = \exp((2\pi i)/5)$ and $v = \exp((2\pi i)/3)$. Clearly $u^i \neq v^j$ as $\gcd(3,5) = 1$.
It is clear from equation (1.12) that any zero of $h - u^i$ (resp. zero of $h - v^j$) is a of order atleast 2 (resp. 2) ($i = 1,2,3,4$). Thus by the second fundamental theorem, we get

$$\begin{aligned} 4T(r,h) &< \sum_{i=1}^{4} \overline{N}(r,u^i;h) + \sum_{j=1}^{2} \overline{N}(r,v^j;h) + S(r,h) \\ &\leq 3T(r,h) + S(r,h), \end{aligned} \tag{1.13}$$

which is a contradiction.
Thus h is a constant function. But as g is a non-constant meromorphic function, so from (1.11), we have

$$(h^5 - 1) \equiv 0 \text{ and } (h^3 - 1) \equiv 0.$$

Thus $h \equiv 1$, i.e., $f \equiv g$. Hence the proof follows. □

Similarly, as *Theorem 1.1*, here we also investigate the case: if multiple zeros of $P_Y(z)$ are taken under consideration, then "*is $P_Y(z)$ a SUPM?*" and if so then "*what about the degree of $P_Y(z)$?*" The next theorem includes all the answers of these questions.

Theorem 1.4. *If $P(z) = z^n + az^{n-r} + b$, where $a \neq 0$, $b \neq 0$, $r \geq 2$, $\gcd(n, n-r) = 1$, then $P(z)$ is a critically injective SUPM (resp. SUPE) for $n \geq r + 5$ (resp.$r + 4$).*

Proof. **[Proof]** Clearly from the given polynomial $P(z)$, we have $P'(z) = z^{n-r-1}(nz^r + (n-r)a)$. So zeros of $P'(z)$ are 0 and c_i for $i = 0,1,\ldots,r-1$. Now we can write

$$c_i = \omega^i \alpha,$$

where ω is the r-th root of unity and $\alpha^r = -\frac{(n-r)a}{n}$. Also

$$P(c_i) = c_i^n + ac_i^{n-r} + b = \omega^{in}\alpha^n + a\omega^{i(n-r)}\alpha^{(n-r)} + b = \omega^{i(n-r)}(\alpha^n + a\alpha^{(n-r)}) + b.$$

As $\gcd(r, n-r) = \gcd(n, n-r) = 1$, so $\omega^{i(n-r)} \neq \omega^{j(n-r)}$ for $i \neq j$, hence $P(c_i) \neq P(c_j)$ for $i \neq j$. Also $P(c_i) \neq P(0)$. Hence $P(z)$ is critically injective.
Let f and g be two non-constant meromorphic functions such that $P(f) = \kappa P(g)$ for $\kappa \in \mathbb{C} \setminus \{0\}$. Thus in view of *Lemma 1.1*, we get

$$T(r,f) = T(r,g) + O(1). \tag{1.14}$$

Now we consider the following cases:
Case-1 Let $\kappa \neq 1$. Then from $P(f) = \kappa P(g)$, we get

$$f^n + af^{(n-r)} + b = \kappa(g^n + ag^{(n-r)} + b)$$

$$\implies f^n + af^{(n-r)} = \kappa\left(g^n + ag^{(n-r)} + b\left(\frac{\kappa - 1}{\kappa}\right)\right). \tag{1.15}$$

Since $b \neq 0$ and $\kappa \neq 1$, then $b(\frac{\kappa-1}{\kappa}) \neq 0$. So we need to discuss the following subcases:
Subcase-1.1 If $b(\frac{\kappa-1}{\kappa}) = -\beta_i$ for any $i \in \{0,1,\ldots,(r-1)\}$, where $\beta_i = c_i^n + ac_i^{(n-r)}$, c_i are the zeros of $f^r + \frac{(n-r)a}{n}$ for $i = 0,1,\ldots,(r-1)$, then (1.15) becomes

$$f^{(n-r)}(f^r + a) = \kappa(g - c_i)^2(g - \zeta_1)...(g - \zeta_{n-2}), \tag{1.16}$$

where ζ_j's are distinct and $\zeta_j \neq c_i$ for $j = 1,2,\ldots,n-2$ and $i = 0,1,\ldots,(r-1)$. Therefore from (1.16), we get

$$\sum_{j=1}^{n-2} \overline{N}(r,\zeta_j;g) + \overline{N}(r,c_i;g) = \overline{N}(r,0;f) + \sum_{j=1}^{r} \overline{N}(r,\varrho_j;f),$$

where ϱ_j's are distinct zeros of $f^r + a$ for $j = 1, 2, \ldots, r$. Now, using the second fundamental theorem and (1.14), we get

$$\begin{aligned}(n-2)T(r,g) &\leq \sum_{j=1}^{n-2} \overline{N}(r,\zeta_j;g) + \overline{N}(r,c_i;g) + \overline{N}(r,\infty;g) + S(r,g) \\ &\leq (r+2)T(r,f) + S(r,g),\end{aligned} \tag{1.17}$$

which is a contradiction for $n \geq r+5$.

Subcase-1.2 If $b(\frac{\kappa-1}{\kappa}) \neq -\beta_i$ for all $i \in \{0, 1, \ldots, (r-1)\}$, then from (1.15) we get

$$f^{(n-r)}(f^r + a) = \kappa(g-\gamma_1)(g-\gamma_2)\ldots(g-\gamma_n), \tag{1.18}$$

where each γ_i are distinct for $i = 1, 2, \ldots, n$. Therefore from (1.18) we get

$$\sum_{i=1}^{n} \overline{N}(r,\gamma_i;g) = \overline{N}(r,0;f) + \sum_{j=1}^{r} \overline{N}(r,\varrho_j;f). \tag{1.19}$$

Henceforth, using (1.19), (1.14) and the second fundamental theorem, we get

$$\begin{aligned}(n-1)T(r,g) &\leq \sum_{i=1}^{n} \overline{N}(r,\gamma_i;g) + \overline{N}(r,\infty;g) + +S(r,g) \\ &\leq (r+2)T(r,f) + S(r,g),\end{aligned} \tag{1.20}$$

which is a contradiction for $n \geq r+5$.

If f and g are entire functions, then from (1.17) and (1.20), we get contradiction for $n \geq r+4$.

Case-2 If $\kappa = 1$, then $P(f) = P(g)$. Thus

$$f^n + af^{(n-r)} + b \equiv g^n + ag^{(n-r)} + b. \tag{1.21}$$

Let $f \not\equiv g$. Then suppose $h = \frac{f}{g}$. Since $a \neq 0$, so from equation (1.21), we get

$$g^r = -a\frac{h^{(n-r)}-1}{h^n-1}. \tag{1.22}$$

Now we consider two subcases:

Subcase-2.1 Let $n - r = 1$. Then we can write (1.22) as,

$$g^{n-1} = -a\frac{1}{(h-\xi_1)\ldots(h-\xi_{n-1})}, \tag{1.23}$$

where $\xi_i^n = 1$ and $\xi_i \neq 1$ for $i = 1, 2, \ldots, n-1$. It is also clear from equation (1.23) that each ξ_i point of h is of multiplicity atleast (n-1). Therefore from the second fundamental theorem, we have

$$\begin{aligned}(n-3)T(r,h) &\leq \sum_{i=1}^{n-1} \overline{N}(r,\xi_i;h) + S(r,h) \\ &\leq (\frac{n-1}{n-1})T(r,h) + S(r,h),\end{aligned} \tag{1.24}$$

which is a contradiction for $n \geq 5$.

Subcase-2.2 Let $n - r \geq 2$. Then again we can write equation (1.22) as

$$g^r = -a\frac{(h-\nu_1)\ldots(h-\nu_{n-r-1})}{(h-\xi_1)...(h-\xi_{n-1})},$$

where ν_i and ξ_i are respectively $(n-r)$-th and n-th root of unity with $\nu_i \neq 1$ and $\xi_i \neq 1$. Also $\nu_i \neq \xi_i$ for each i as $\gcd(n, n-r) = 1$. Also each ξ_i and ν_i point of h is of multiplicity atleast 2 because $r \geq 2$. Thus using the second fundamental theorem, we get

$$\begin{aligned}(n-2)T(r,h) &\leq \sum_{i=1}^{n-1} \overline{N}(r,\xi_i;h) + \overline{N}(r,\nu_i;h) + S(r,h) \qquad (1.25)\\ &\leq (\frac{n-1}{2})T(r,h) + \frac{1}{2}T(r,h) + S(r,h),\end{aligned}$$

which is a contradiction for $n \geq 5$.

But if f and g are entire functions, then

$$\overline{N}(r,\xi_i;h) \leq \overline{N}(r,g) = S(r,g) = S(r,h).$$

Hence we get a contradiction from (1.24) and (1.25) for $n \geq 4$. Therefore $f \equiv g$. Hence the proof. □

Remark 1.4. *i) If* $\gcd(n, n-r) \neq 1$ *in the above Theorem 1.4, then there exist many polynomials of the form* $P(z) = z^n + az^{n-r} + b$, *where* $\gcd(n, n-r) = d > 1$, *but* $P(f) = P(\sigma f)$, *where* σ *is the non-real d'th root of unity. Therefore* $P(z)$ *is not a UPM at all. So* $\gcd(n, n-r) = 1$ *is essential for* $P(z)$ *to be a SUPM.*

ii) If $r = 1$, *then for any any non-constant meromorphic function h, we set*

$$f = -a\frac{h^{n-1}}{1-h^n}, \quad g = -ah\frac{h^{n-1}}{1-h^n}.$$

Then

$$\left(\frac{f}{g}\right)^{n-1}\left(\frac{f+a}{g+a}\right) = \frac{1}{h^{n-1}}\frac{h^{n-1}-h^n}{1-h} = 1.$$

That is, $P(f) = P(g)$ *holds for any* $b \in \mathbb{C}$ *but* $f = hg$. *Obviously* $P(z)$ *is not a UPM. Therefore* $r \geq 2$ *is also essential for* $P(z)$ *to be SUPM.*

iii) Now it comes to $ab \neq 0$, *which is sufficient for* $P(z) = z^n + az^{n-r} + b$ *to be SUPM, so it is inevitable to ask what will happen if* $ab = 0$ *? If both a and b becomes zero, then it is obvious that* $P(z)$ *is not a SUPM. If* $ab = 0$ *and* $a + b \neq 0$ *then the following three theorem will tell us that* $ab \neq 0$ *is not only sufficient but also necessary for* $P(z)$ *to be SUPM.*

If $a = 0$, then we have the following result:

Theorem 1.5. *Suppose* $P(z) = z^n + b$, *where* $b \neq 0$. *If for any two non-constant meromorphic functions f and g,* $P(f) = \kappa P(g)$ *holds, then* $f = \omega g$, *where* ω *is the n-th root of unity for* $n \geq 4$.

Proof. **[Proof]** In this case, $(f^n + b) = \kappa(g^n + b)$.
Assume $\kappa \neq 1$, then by putting $f = gh$, we get

$$g^n(h^n - \kappa) = b(\kappa - 1).$$

As g is non-constant meromorphic function and $\kappa \neq 1$, so h is a non-constant meromorphic function. It is also clear that each ξ_i'-pt of h (where $(\xi_i')^n = \kappa$) is a pole of g and hence ξ_i'-pt of h is of multiplicity atleast n. Thus applying the second fundamental theorem, we have

$$\begin{aligned}(n-2)T(r,h) &\leq \sum_{i=1}^{n} \overline{N}(r,\xi_i';h) + S(r,h)\\ &\leq T(r,h) + S(r,h),\end{aligned}$$

which is a contradiction as $n \geq 4$.

Thus $\kappa = 1$, hence $f = \omega g$, where ω is the n-th root of unity. Hence the proof. □

Next we assume that $b = 0$. To deal this case, we need to initiate the following definition:

Definition 1.7. *If for a non-constant polynomial $P(z)$ in $\mathbb{C}$ and for two non-constant meromorphic (resp. entire) functions f and g, $P(f) \equiv \kappa P(g)$ implies $f = g$ for any non-zero complex constant $\kappa \in \mathbb{C} \backslash B$, where $B \subset \mathbb{C}$ is some finite set, then the polynomial $P(z)$ in $\mathbb{C}$ is called a* **restricted strong uniqueness polynomial for meromorphic (resp. entire) functions over** $\overline{B}$, *where $\overline{B} = \mathbb{C} \setminus B$. We say $P(z)$ is a $RSUPM_{\overline{B}}$ (resp. $RSUPE_{\overline{B}}$) in brief.*

If we take $B = \phi$ in the above definition, we get the usual definition of SUPM (resp. SUPE).

Theorem 1.6. *Suppose $P(z) = z^n + az^{n-r}$, where $\gcd(n, r) = 1$ and $r \geq 2$. If $a \neq 0$, then $P(z)$ is a RSUPM (resp. RSUPE) of degree $n \geq 5$ (resp. 4) over $\overline{B}$, where $B = \{z \mid z^r = 1 \text{ but } z \neq 1\}$.*

Proof. **[Proof]** Obviously $P(z)$ is critically injective. Now suppose for any two non-constant meromorphic functions f and g and a non-zero constant $\kappa \in \mathbb{C} \setminus B$, $P(f) \equiv \kappa P(g)$ holds. Then

$$(f^n - \kappa g^n) \equiv -a(f^{n-r} - \kappa g^{n-r}). \tag{1.26}$$

Putting $f = gh$ we get

$$g^r(h^n - \kappa) \equiv -a(h^{n-r} - \kappa). \tag{1.27}$$

Then

$$T(r, h) = O(T(r, g)) \text{ and } S(r, h) = S(r, g). \tag{1.28}$$

Now we consider two cases:

Case-1 Assume $f \not\equiv g$. Then $\kappa \neq 1$, otherwise proceeding same as Case-2 of *Theorem 1.4*, we get $f \equiv g$, which is impossible. Also we have

$$g^r(h^n - \kappa) \equiv -a(h^{n-r} - \kappa). \tag{1.29}$$

Assume that h is a constant function. Then as g is non-constant meromorphic function, so we have

$$(h^n - \kappa) = 0 \text{ and } (h^{n-r} - \kappa) = 0.$$

Thus $h^r = 1$. But $h \neq 1$ as $\kappa \neq 1$, so $h = \omega$, where $\omega(\neq 1)$ is the r-th root of unity. Hence $\kappa \in B$, which is again a contradiction.
Thus h is a non-constant meromorphic function. Now we can write equation (1.29) as

$$g^r \equiv -a\frac{(h - \nu_1')(h - \nu_2')...(h - \nu_{n-r}')}{(h - \xi_1')(h - \xi_2')...(h - \xi_n')}, \tag{1.30}$$

where ν_j's are the zeros of $(h^{n-r} - \kappa)$ for $i = 1, 2, \ldots, n - r$ and ξ_i's are the zeros of $(h^n - \kappa)$, for $j = 1, 2, \ldots, n$. Here $h^{n-r} - \kappa = 0$ and $h^n - \kappa = 0$ has no common zero, otherwise, if α be a common zero, then $\alpha^n = \alpha^{n-r} = \kappa$, i.e., $\alpha^r = 1$, which gives $\alpha = 1$ or ω^i for $i = 1, 2, \ldots, (r - 1)$ and hence $\kappa = 1$ or ω^i for $i = 1, 2, \ldots, r - 1$, which is a contradiction.

Therefore applying the same technique as in Case-2 of *Theorem 1.4* we get a contradiction for $n \geq 5$ (resp. $n \geq 4$).

Case-2 Next we assume that $f \equiv g$.
Hence the proof. □

In view of *Theorem 1.6*, the natural inquisition would be to study the situation when the constant κ run over the whole complex plane $\mathbb{C}$.

In the following theorem, we can settle this problem.

Theorem 1.7. *Suppose $P(z) = z^n + az^{n-r}$, where $\gcd(n, r) = 1$ and $r \geq 2$ and $a \neq 0$. If for any two non-constant meromorphic (resp. entire) functions f and g, $P(f) = \kappa P(g)$ holds, then $f = \omega g$, where ω is the r-th root of unity and $n \geq 5$ (resp. 4) is an integer.*

Proof. **[Proof]** In this theorem, we consider two cases:

Case-1 If $\kappa \in \overline{B}$, then by *Theorem 1.6*, we get $f \equiv g$.

Case-2 Let $\kappa \in B$, i.e., $\kappa \neq 1$ and $\kappa = \omega^i$ (i = 1,2,...,(r-1)), where $\omega(\neq 1)$ and $\omega^r = 1$.

Now as $\gcd(n, n-r) = 1$, so $\gcd(n, r) = 1$ and $\gcd(r, n-r) = 1$. Thus there exist integers p, q, s, t such that $ns + rt = 1$ and $(n-r)p + rq = 1$. Thus

$$\omega^i = (\omega^{ns+rt})^i = [\{\omega\}^{ns}\{\omega^r\}^t]^i = (\omega^{si})^n,$$

and

$$\omega^i = (\omega^{(n-r)p+rq})^i = [\{\omega\}^{(n-r)p}\{\omega^r\}^q]^i = (\omega^{pi})^{(n-r)}.$$

Also,

$$\omega^{pi} = (\omega^i)^p = ((\omega^{si})^n)^p = (\omega^{psi})^n = (\omega^{psi})^{n-r}(\omega^{psi})^r = (\omega^{psi})^{(n-r)} = ((\omega^{pi})^{n-r})^s = \omega^{si}.$$

By the given condition, here we can also get equation (1.27), i.e.,

$$g^r(h^n - \kappa) = -a(h^{n-r} - \kappa) \tag{1.31}$$

Next we consider two subcases:

Subcase-2.1 Suppose that h is a constant function. Then as g is non-constant, so from (1.31), we have

$$(h^n - \kappa) = 0 \ \text{ and } \ (h^{n-r} - \kappa) = 0.$$

Thus $h^r = 1$. If $h = 1$, then $\kappa = 1$, which is contradiction. So $h = \omega$, where ω $(\neq 1)$ is the r-th root of unity. Hence $f = \omega g$.

Subcase-2.2 Assume that h is a non-constant meromorphic function.

In this case, equation (1.31) can be written as

$$\begin{aligned} g^r &= -a\frac{h^{n-r} - \omega^i}{h^n - \omega^i} \\ &= -a\frac{h^{n-r} - (\omega^{pi})^{n-r}}{h^n - (\omega^{si})^n} \\ &= -a\frac{h^{n-r} - (\omega^{pi})^{n-r}}{h^n - (\omega^{pi})^n} \end{aligned}$$

Hence

$$g^r = -a\frac{(h - \omega^{pi})\prod_{j=1}^{n-r-1}(h - \lambda_j)}{(h - \omega^{pi})\prod_{t=1}^{n-1}(h - \mu_t)}, \tag{1.32}$$

where λ_j and μ_t are distinct zeros of $h^{n-r} - \omega^i$ and $h^n - \omega^i$ respectively. Also $\lambda_j \neq \mu_t$.

We omit rest of the proof of this theorem since same can be dealt as in the line of proof of Case-2 of *Theorem 1.4*.

This completes the proof. □

Now for all the above and existing results of SUPM (resp. SUPE), it is ineluctable to ask whether the strong uniqueness polynomials can further be generalized with any linear transformation? Next two theorems serve us the answer.

Theorem 1.8. *Let P_1 be a SUPM. Then $P_1 o P_2$ is a SUPM if and only if P_2 is a UPM.*

Proof. **[Proof]** Let us assume that for any two non-constant meromorphic functions f and g and a non-zero constant $\kappa \in \mathbb{C}$, $(P_1 o P_2)(f) = \kappa (P_1 o P_2)(g)$. Then clearly $f = g$ as P_1 is a SUPM and P_2 is a UPM.

Conversely, for any two non-constant meromorphic functions f and g, suppose $P_2(f) = P_2(g)$. Then $(P_1 o P_2)(f) = (P_1 o P_2)(g)$, which implies $f = g$ as $P_1 o P_2$ is a SUPM. □

Corollary 1.1. *If $P(z)$ is a SUPM (resp. SUPE), then $P(az+b)$ is also SUPM (resp. SUPE) for any non-zero complex constant a.*

Proof. **[Proof]** As $P_2(z) = az + b$ is a UPM for any non-zero complex constant a and $P(z)$ is a SUPM, then by the above theorem we have PoP_2 is a SUPM, i.e., $P(az + b)$ is a SUPM. □

After the vivid discussion of critically injective SUPM (resp. SUPE) for ITGP we are concluding the paper with the following ineluctable question.

Question 1.2. *Does there exist any critically injective* SUPM *of degree* 5 *having multiple zeros?*

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