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Reverse Order Law $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ in Rings with Involution

Dijana Mosić[a], Nebojša Č. Dinčić[a]

[a] *Faculty of Sciences and Mathematics, University of Niš, P.O. Box 224, 18000 Niš, Serbia*

Abstract. In this paper we study several equivalent conditions for the reverse order law $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ in rings with involution. We extend some well-known results to more general settings.

1. Introduction

Let $\mathcal{R}$ be an associative ring with the unit 1. If $a, b \in \mathcal{R}$ are invertible, then ab is invertible too and the inverse of the product ab satisfied the reverse order law $(ab)^{-1} = b^{-1}a^{-1}$. This formula cannot trivially be extended to the Moore–Penrose inverse of the product ab. Many authors studied this problem and proved some equivalent conditions for $(ab)^\dagger = b^\dagger a^\dagger$ in setting of matrices, operators or rings [1–6, 8, 9, 11, 12, 15, 20–22]. Because the reverse order law $(ab)^\dagger = b^\dagger a^\dagger$ does not always holds, it is not easy to simplify various expressions that involve the Moore-Penrose inverse of product. In addition to $(ab)^\dagger = b^\dagger a^\dagger$, $(ab)^\dagger$ may be expressed as $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$, $(ab)^\dagger = b^*(a^*abb^*)^\dagger a^*$, $(ab)^\dagger = b^\dagger a^\dagger - b^\dagger[(1-bb^\dagger)(1-a^\dagger a)]^\dagger a^\dagger$ etc. These equalities are called mixed-type reverse order laws for the Moore-Penrose inverse of a product. When investigating various reverse order laws for $(ab)^\dagger$, we notice that some of them are in fact equivalent (see [15, 19, 20]). In this paper we investigate necessary and sufficient conditions for the reverse order law $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ in the setting of rings with involution.

An involution $a \mapsto a^*$ in a ring $\mathcal{R}$ is an anti-isomorphism of degree 2, that is,

$$(a^*)^* = a, \quad (a+b)^* = a^* + b^*, \quad (ab)^* = b^*a^*.$$

An element $a \in \mathcal{R}$ is self-adjoint if $a^* = a$.

The *Moore–Penrose inverse* (or *MP-inverse*) of $a \in \mathcal{R}$ is the element $b \in \mathcal{R}$, if the following equations hold [16–18]:

$$(1)\ aba = a, \quad (2)\ bab = b, \quad (3)\ (ab)^* = ab, \quad (4)\ (ba)^* = ba.$$

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Email addresses: dijana@pmf.ni.ac.rs (Dijana Mosić), ndincic@hotmail.com (Nebojša Č. Dinčić)

There is at most one b such that above conditions hold (see [17]), and such b is denoted by $a^\dagger$. The set of all Moore–Penrose invertible elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^\dagger$. If a is invertible, then $a^\dagger$ coincides with the ordinary inverse of a.

If $\delta \subset \{1,2,3,4\}$ and b satisfies the equations (i) for all $i \in \delta$, then b is an δ–inverse of a. The set of all δ–inverse of a is denote by $a\{\delta\}$. Notice that $a\{1,2,3,4\} = \{a^\dagger\}$.

The following result is well-known and frequently used in the rest of the paper.

Theorem 1.1. [7, 14] *For any $a \in \mathcal{R}^\dagger$, the following is satisfied:*

(a) $(a^\dagger)^\dagger = a$;

(b) $(a^*)^\dagger = (a^\dagger)^*$;

(c) $(a^*a)^\dagger = a^\dagger(a^\dagger)^*$;

(d) $(aa^*)^\dagger = (a^\dagger)^*a^\dagger$;

(f) $a^* = a^\dagger aa^* = a^*aa^\dagger$;

(g) $a^\dagger = (a^*a)^\dagger a^* = a^*(aa^*)^\dagger$;

(h) $(a^*)^\dagger = a(a^*a)^\dagger = (aa^*)^\dagger a$.

From the last theorem we see that the following chain of equivalences hold:

$$a \in \mathcal{R}^\dagger \Leftrightarrow a^* \in \mathcal{R}^\dagger \Leftrightarrow aa^* \in \mathcal{R}^\dagger \Leftrightarrow a^*a \in \mathcal{R}^\dagger.$$

Let $\mathcal{A}$ be a unital C^*–algebra with the unit 1. An element $a \in \mathcal{A}$ is regular if there exists some $b \in \mathcal{A}$ satisfying $aba = a$.

Theorem 1.2. [10] *In a unital C^*–algebra $\mathcal{A}$, $a \in \mathcal{A}$ is MP-invertible if and only if a is regular.*

An element $p \in \mathcal{A}$ is a projection if $p = p^2 = p^*$. Set $\mathcal{P}(\mathcal{A}) = \{p \in \mathcal{A} : p^2 = p = p^*\}$. In [13], Li proved the following important results which consider some equivalent conditions for pq, $(p, q \in \mathcal{P}(\mathcal{A}))$, to be Moore-Penrose invertible and formula for Moore-Penrose inverse of product of projection in a C^*–algebra.

Lemma 1.3. [13] *Let $p, q \in \mathcal{P}(\mathcal{A})$. Then the following statements are equivalent:*

(a) *pq is Moore-Penrose invertible;*

(b) *qp is Moore-Penrose invertible;*

(c) *$(1-p)(1-q)$ is Moore-Penrose invertible;*

(d) *$(1-q)(1-p)$ is Moore-Penrose invertible.*

Theorem 1.4. [13] *Let $p, q \in \mathcal{P}(\mathcal{A})$. If pq is Moore-Penrose invertible, then:*

$$(qp)^\dagger = pq - p[(1-p)(1-q)]^\dagger q.$$

The reverse order law for the Moore-Penrose inverse is an useful computational tool in applications (solving linear equations in linear algebra or numerical analysis), and it is also interesting from the theoretical point of view.

The reverse-order law $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ was first studied by Galperin and Waksman [8]. A Hilbert space version of their result was given by Isumino [11]. They proved that $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ holds if and only if $\mathcal{R}((a^*)^\dagger b) = \mathcal{R}(ab)$ and $\mathcal{R}(b^\dagger a^*) = \mathcal{R}((ab)^*)$, for linear operators a and b, where $\mathcal{R}(\cdot)$ denotes the range of an operator. Many results concerning the reverse order law $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ for complex matrices appeared in Tian's papers [19] and [20], where the author used finite dimensional methods (mostly properties of the rank of a complex matrices). Moreover, the operator analogues of these results are proved in [4] for linear operators on Hilbert spaces, using the operator matrices. In [15], a set of equivalent conditions for this reverse order rule for the Moore-Penrose inverse in the setting of C^*-algebra is presented, extending the results for complex matrices from [20]. This result can be formulate for elements in ring with involution in the following way.

Theorem 1.5. [15] *Let $\mathcal{R}$ be a ring with involution and let $a, b \in \mathcal{R}^\dagger$. Then the following statements are equivalent:*

(a) $ab, a^\dagger abb^\dagger \in \mathcal{R}^\dagger$ *and* $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$;

(b) $ab, a^\dagger abb^\dagger \in \mathcal{R}^\dagger$ *and* $(a^\dagger abb^\dagger)^\dagger = b(ab)^\dagger a$;

(c) $ab, a^\dagger ab, abb^\dagger \in \mathcal{R}^\dagger$ *and* $(ab)^\dagger = (a^\dagger ab)^\dagger a^\dagger = b^\dagger(abb^\dagger)^\dagger$;

(d) $ab, a^\dagger ab, abb^\dagger \in \mathcal{R}^\dagger$, $(a^\dagger ab)^\dagger = (ab)^\dagger a$ *and* $(abb^\dagger)^\dagger = b(ab)^\dagger$;

(e) $a^\dagger ab, abb^\dagger, a^\dagger abb^\dagger \in \mathcal{R}^\dagger$, $(a^\dagger ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger$ *and* $(abb^\dagger)^\dagger = (a^\dagger abb^\dagger)^\dagger a^\dagger$;

(f) $ab, a^*abb^* \in \mathcal{R}^\dagger$ *and* $(ab)^\dagger = b^*(a^*abb^*)^\dagger a^*$;

(g) $ab, a^*abb^* \in \mathcal{R}^\dagger$ *and* $(a^*abb^*)^\dagger = (b^*)^\dagger(ab)^\dagger(a^*)^\dagger$.

In this paper we present new results for the reverse order law $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ in rings with involution. Thus, we extend the known results for matrices [19] and for Hilbert space operators [4] to more general settings. The most important properties of the MP-inverse will be used in proving various equivalent conditions such that the reverse order law $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ holds. Although these results are known, we use different methods, depending on algebraic properties of rings with involution.

2. Reverse Order Law in Rings

In this section we present necessary and sufficient conditions such that the reverse order law $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ holds.

Theorem 2.1. *Let $\mathcal{R}$ be a ring with involution and let $a, b \in \mathcal{R}^\dagger$. Then the following statements are equivalent:*

(a1) $ab, a^\dagger abb^\dagger \in \mathcal{R}^\dagger$ *and* $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$;

(a2) $ab, a^*abb^* \in \mathcal{R}^\dagger$ *and* $(ab)^\dagger = b^*(a^*abb^*)^\dagger a^*$;

(b1) $(a^\dagger)^*b, a^\dagger abb^\dagger \in \mathcal{R}^\dagger$ *and* $[(a^\dagger)^*b]^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^*$;

(b2) $(a^\dagger)^*b, (a^*a)^\dagger bb^* \in \mathcal{R}^\dagger$ *and* $[(a^\dagger)^*b]^\dagger = b^*[(a^*a)^\dagger bb^*]^\dagger a^\dagger$;

(c1) $a(b^\dagger)^*, a^\dagger abb^\dagger \in \mathcal{R}^\dagger$ *and* $[a(b^\dagger)^*]^\dagger = b^*(a^\dagger abb^\dagger)^\dagger a^\dagger$;

(c2) $a(b^\dagger)^*, a^*a(bb^*)^\dagger \in \mathcal{R}^\dagger$ *and* $[a(b^\dagger)^*]^\dagger = b^\dagger[a^*a(bb^*)^\dagger]^\dagger a^*$;

(d1) $b^\dagger a^\dagger, bb^\dagger a^\dagger a \in \mathcal{R}^\dagger$ *and* $(b^\dagger a^\dagger)^\dagger = a(bb^\dagger a^\dagger a)^\dagger b$;

(d2) $b^\dagger a^\dagger, (bb^*)^\dagger(a^*a)^\dagger \in \mathcal{R}^\dagger$ *and* $(b^\dagger a^\dagger)^\dagger = (a^\dagger)^*[(bb^*)^\dagger(a^*a)^\dagger]^\dagger(b^\dagger)^*$;

(e1) $a^\dagger ab, abb^\dagger \in \mathcal{R}^\dagger$ *and* $(a^\dagger ab)^\dagger a^\dagger = b^\dagger(abb^\dagger)^\dagger$;

(e2) $a^\dagger ab, (a^\dagger)^*bb^\dagger \in \mathcal{R}^\dagger$ *and* $(a^\dagger ab)^\dagger a^* = b^\dagger[(a^\dagger)^*bb^\dagger]^\dagger$;

(e3) $a^\dagger a(b^\dagger)^*, abb^\dagger \in \mathcal{R}^\dagger$ *and* $[a^\dagger a(b^\dagger)^*]^\dagger a^\dagger = b^*(abb^\dagger)^\dagger$;

(e4) $bb^\dagger a^\dagger, b^\dagger a^\dagger a \in \mathcal{R}^\dagger$ *and* $(bb^\dagger a^\dagger)^\dagger b = a(b^\dagger a^\dagger a)^\dagger$;

(e5) $a^*ab, abb^* \in \mathcal{R}^\dagger$ *and* $(a^*ab)^\dagger a^* = b^*(abb^*)^\dagger$;

(e6) $(a^*a)^\dagger b, (a^\dagger)^*bb^* \in \mathcal{R}^\dagger$ *and* $[(a^*a)^\dagger b]^\dagger a^\dagger = b^*[(a^\dagger)^*bb^*]^\dagger$;

(e7) $a^*a(b^\dagger)^*, a(bb^*)^\dagger \in \mathcal{R}^\dagger$ *and* $[a^*a(b^\dagger)^*]^\dagger a^* = b^\dagger[a(bb^*)^\dagger]^\dagger$;

(e8) $(a^\dagger)^*(bb^*)^\dagger, (a^*a)^\dagger(b^\dagger)^* \in \mathcal{R}^\dagger$ *and* $b^\dagger[(a^\dagger)^*(bb^*)^\dagger]^\dagger = [(a^*a)^\dagger(b^\dagger)^*]^\dagger a^\dagger$;

(e9) $aa^*abb^*b, a^*abb^* \in \mathcal{R}^\dagger$ *and* $(aa^*abb^*b)^\dagger = b^\dagger(a^*abb^*)^\dagger a^\dagger$;

(f1) $a^\dagger ab, abb^\dagger, a^\dagger abb^\dagger \in \mathcal{R}^\dagger$, $(a^\dagger ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger$ *and* $(abb^\dagger)^\dagger = (a^\dagger abb^\dagger)^\dagger a^\dagger$;

(f2) $a^\dagger ab, abb^\dagger, a^\dagger abb^*, a^*abb^\dagger \in \mathcal{R}^\dagger$, $(a^\dagger ab)^\dagger = b^*(a^\dagger abb^*)^\dagger$ *and* $(abb^\dagger)^\dagger = (a^*abb^\dagger)^\dagger a^*$.

Proof. The equivalences (a1) ⇔ (a2) ⇔ (f1) follow from Theorem 1.5.

(a1) ⇒ (b1): Using the hypothesis $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ and Theorem 1.1, we get

$$(a^\dagger)^* bb^\dagger(a^\dagger abb^\dagger)^\dagger a^*(a^\dagger)^* b = (a^\dagger)^* a^\dagger(abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger ab) = (a^\dagger)^* a^\dagger ab = (a^\dagger)^* b,$$

$$\begin{aligned} b^\dagger(a^\dagger abb^\dagger)^\dagger a^*(a^\dagger)^* bb^\dagger(a^\dagger abb^\dagger)^\dagger a^* &= (b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger)aa^* = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger aa^* \\ &= b^\dagger(a^\dagger abb^\dagger)^\dagger a^*, \end{aligned}$$

$$\begin{aligned} ((a^\dagger)^* bb^\dagger(a^\dagger abb^\dagger)^\dagger a^*)^* &= ((a^\dagger)^* a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^*)^* = aa^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger \\ &= abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger = (abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger)^* \\ &= (aa^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger)^* = (a^\dagger)^* a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^* \\ &= (a^\dagger)^* bb^\dagger(a^\dagger abb^\dagger)^\dagger a^*, \end{aligned}$$

$$(b^\dagger(a^\dagger abb^\dagger)^\dagger a^*(a^\dagger)^* b)^* = (b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger ab)^* = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger ab = b^\dagger(a^\dagger abb^\dagger)^\dagger a^*(a^\dagger)^* b.$$

Hence, by these four equalities and the definition of MP-inverse, we deduce that $(a^\dagger)^* b \in \mathcal{R}^\dagger$ and $[(a^\dagger)^* b]^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^*$.

(b1) ⇒ (a1): Since

$$abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger ab = a(a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger)b = aa^\dagger abb^\dagger b = ab, \tag{1}$$

$$b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger, \tag{2}$$

we conclude that $b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger \in (ab)\{1,2\}$. From $[(a^\dagger)^* b]^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^*$, we have that the elements $(a^\dagger)^* bb^\dagger(a^\dagger abb^\dagger)^\dagger a^*$, $b^\dagger(a^\dagger abb^\dagger)^\dagger a^*(a^\dagger)^* b$ are self-adjoin. Then

$$\begin{aligned} (abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger)^* &= (aa^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger)^* = (a^\dagger)^* a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^* \\ &= (a^\dagger)^* bb^\dagger(a^\dagger abb^\dagger)^\dagger a^* = ((a^\dagger)^* bb^\dagger(a^\dagger abb^\dagger)^\dagger a^*)^* \\ &= ((a^\dagger)^* a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^*)^* = aa^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger \\ &= abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger, \end{aligned}$$

$$\begin{aligned} (b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger ab)^* &= (b^\dagger(a^\dagger abb^\dagger)^\dagger a^*(a^\dagger)^* b)^* = b^\dagger(a^\dagger abb^\dagger)^\dagger a^*(a^\dagger)^* b \\ &= b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger ab, \end{aligned}$$

i.e. $abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$, $b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger ab$ are self-adjoin too. Therefore, $ab \in \mathcal{R}^\dagger$ and $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$.

(a1) ⇒ (c1): By the definition of MP-inverse and Theorem 1.1, we obtain

$$a(b^\dagger)^* b^*(a^\dagger abb^\dagger)^\dagger a^\dagger a(b^\dagger)^* = a(a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger)(b^\dagger)^* = aa^\dagger abb^\dagger(b^\dagger)^* = a(b^\dagger)^*,$$

$$b^*(a^\dagger abb^\dagger)^\dagger a^\dagger a(b^\dagger)^* b^*(a^\dagger abb^\dagger)^\dagger a^\dagger = b^*(a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger = b^*(a^\dagger abb^\dagger)^\dagger a^\dagger,$$

i.e. $b^*(a^\dagger abb^\dagger)^\dagger a^\dagger \in [a(b^\dagger)^*]\{1,2\}$. The condition $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ give that the right hand side of the equality

$$a(b^\dagger)^* b^*(a^\dagger abb^\dagger)^\dagger a^\dagger = abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$$

is self-adjoint element. So, $a(b^\dagger)^* b^*(a^\dagger abb^\dagger)^\dagger a^\dagger$ is self-adjoint too. In the same way, from the equality

$$(b^*(a^\dagger abb^\dagger)^\dagger a^\dagger a(b^\dagger)^*)^* = (b^*(a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger(b^\dagger)^*)^* = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger b = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger ab,$$

we conclude that $b^*(a^\dagger abb^\dagger)^\dagger a^\dagger a(b^\dagger)^*$ is self-adjoint. Hence, $a(b^\dagger)^* \in \mathcal{R}^\dagger$ and $[a(b^\dagger)^*]^\dagger = b^*(a^\dagger abb^\dagger)^\dagger a^\dagger$.

(c1) $\Rightarrow$ (a1): By (1) and (2), we have $b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger \in (ab)\{1,2\}$. Since $[a(b^\dagger)^*]^\dagger = b^*(a^\dagger abb^\dagger)^\dagger a^\dagger$, then $a(b^\dagger)^* b^*(a^\dagger abb^\dagger)^\dagger a^\dagger, b^*(a^\dagger abb^\dagger)^\dagger a^\dagger a(b^\dagger)^*$ are self-adjoin. Thus, from

$$abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger = a(b^\dagger)^* b^*(a^\dagger abb^\dagger)^\dagger a^\dagger,$$

$$(b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger ab)^* = (b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger b)^* = b^*(a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger(b^\dagger)^* = b^*(a^\dagger abb^\dagger)^\dagger a^\dagger a(b^\dagger)^*,$$

we deduce that the elements $abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger, b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger ab$ are self-adjoin too. So, we get that $ab \in \mathcal{R}^\dagger$ and $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$, i.e. the condition (a1) is satisfied.

(a1) $\Rightarrow$ (d1): The condition $a^\dagger abb^\dagger \in \mathcal{R}^\dagger$, by Theorem 1.1, implies $bb^\dagger a^\dagger a = (a^\dagger abb^\dagger)^* \in \mathcal{R}^\dagger$. Now we prove that $a(bb^\dagger a^\dagger a)^\dagger b \in (b^\dagger a^\dagger)\{1,2\}$:

$$b^\dagger a^\dagger a(bb^\dagger a^\dagger a)^\dagger bb^\dagger a^\dagger = b^\dagger(bb^\dagger a^\dagger a(bb^\dagger a^\dagger a)^\dagger bb^\dagger a^\dagger a)a^\dagger = b^\dagger bb^\dagger a^\dagger aa^\dagger = b^\dagger a^\dagger,$$

$$a(bb^\dagger a^\dagger a)^\dagger bb^\dagger a^\dagger a(bb^\dagger a^\dagger a)^\dagger b = a(bb^\dagger a^\dagger a)^\dagger b.$$

Further, by (a1) $\Leftrightarrow$ (c1) and the equality

$$(b^\dagger a^\dagger a(bb^\dagger a^\dagger a)^\dagger b)^* = b^*[(bb^\dagger a^\dagger a)^*]^\dagger a^\dagger a(b^\dagger)^* = b^*(a^\dagger abb^\dagger)^\dagger a^\dagger a(b^\dagger)^*,$$

it follows that the element $b^\dagger a^\dagger a(bb^\dagger a^\dagger a)^\dagger b$ is self-adjoint. To conclude that $a(bb^\dagger a^\dagger a)^\dagger bb^\dagger a^\dagger$ is self-adjoint, we consider the equivalence (a1) $\Leftrightarrow$ (b1) and the equality

$$(a(bb^\dagger a^\dagger a)^\dagger bb^\dagger a^\dagger)^* = (a^\dagger)^* bb^\dagger[(bb^\dagger a^\dagger a)^*]^\dagger a^* = (a^\dagger)^* bb^\dagger(a^\dagger abb^\dagger)^\dagger a^*.$$

Therefore, $b^\dagger a^\dagger \in \mathcal{R}^\dagger$ and $(b^\dagger a^\dagger)^\dagger = a(bb^\dagger a^\dagger a)^\dagger b$.

(d1) $\Rightarrow$ (a1): We observe that by (1) and (2), $b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger \in (ab)\{1,2\}$. If the hypothesis $(b^\dagger a^\dagger)^\dagger = a(bb^\dagger a^\dagger a)^\dagger b$ holds, the elements $b^\dagger a^\dagger a(bb^\dagger a^\dagger a)^\dagger b$ and $a(bb^\dagger a^\dagger a)^\dagger bb^\dagger a^\dagger$ are self-adjoint. Then, from

$$\begin{aligned} abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger &= ((a^\dagger)^*[(a^\dagger abb^\dagger)^*]^\dagger bb^\dagger a^*)^* = ((a^\dagger)^*(bb^\dagger a^\dagger a)^\dagger bb^\dagger a^\dagger aa^*)^* \\ &= a(bb^\dagger a^\dagger a)^\dagger bb^\dagger a^\dagger aa^\dagger = a(bb^\dagger a^\dagger a)^\dagger bb^\dagger a^\dagger \end{aligned}$$

and

$$\begin{aligned} b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger ab &= (b^* a^\dagger a[(a^\dagger abb^\dagger)^*]^\dagger(b^\dagger)^*)^* = (b^* bb^\dagger a^\dagger a(bb^\dagger a^\dagger a)^\dagger(b^\dagger)^*)^* \\ &= b^\dagger bb^\dagger a^\dagger a(bb^\dagger a^\dagger a)^\dagger b = b^\dagger a^\dagger a(bb^\dagger a^\dagger a)^\dagger b, \end{aligned}$$

we have $b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger \in (ab)\{3,4\}$. So, $ab \in \mathcal{R}^\dagger$ and $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$.

(b1) ⇒ (b2): First we will prove that $(a^*a)^\dagger bb^* = a^\dagger(a^\dagger)^*bb^* \in \mathcal{R}^\dagger$ and $(a^\dagger(a^\dagger)^*bb^*)^\dagger = (b^\dagger)^*[(a^\dagger)^*b]^\dagger a$. Indeed, the equalities

$$a^\dagger(a^\dagger)^*bb^*(b^\dagger)^*[(a^\dagger)^*b]^\dagger aa^\dagger(a^\dagger)^*bb^* = a^\dagger((a^\dagger)^*b[(a^\dagger)^*b]^\dagger(a^\dagger)^*b)b^* = a^\dagger(a^\dagger)^*bb^* \tag{3}$$

and

$$\begin{aligned}(b^\dagger)^*[(a^\dagger)^*b]^\dagger aa^\dagger(a^\dagger)^*bb^*(b^\dagger)^*[(a^\dagger)^*b]^\dagger a &= (b^\dagger)^*[(a^\dagger)^*b]^\dagger(a^\dagger)^*b[(a^\dagger)^*b]^\dagger a \\ &= (b^\dagger)^*[(a^\dagger)^*b]^\dagger a \end{aligned}\tag{4}$$

imply that $(b^\dagger)^*[(a^\dagger)^*b]^\dagger a \in (a^\dagger(a^\dagger)^*bb^*)\{1,2\}$. The assumption $[(a^\dagger)^*b]^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^*$ gives

$$\begin{aligned}a^\dagger(a^\dagger)^*bb^*(b^\dagger)^*[(a^\dagger)^*b]^\dagger a &= a^\dagger(a^\dagger)^*b[(a^\dagger)^*b]^\dagger a = (a^*(a^\dagger)^*b[(a^\dagger)^*b]^\dagger(a^\dagger)^*)^* \\ &= (a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^*(a^\dagger)^*)^* \\ &= a^\dagger aa^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger = a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger\end{aligned}$$

and

$$\begin{aligned}(b^\dagger)^*[(a^\dagger)^*b]^\dagger aa^\dagger(a^\dagger)^*bb^* &= (b^\dagger)^*[(a^\dagger)^*b]^\dagger(a^\dagger)^*bb^* = (b[(a^\dagger)^*b]^\dagger(a^\dagger)^*bb^\dagger)^* \\ &= (bb^\dagger(a^\dagger abb^\dagger)^\dagger a^*(a^\dagger)^*bb^\dagger)^* \\ &= (bb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger)^* \\ &= (a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger bb^\dagger = (a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger.\end{aligned}$$

Since $a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger$ and $(a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger$ are self-adjoint, it follows that $a^\dagger(a^\dagger)^*bb^*(b^\dagger)^*[(a^\dagger)^*b]^\dagger a$ and $(b^\dagger)^*[(a^\dagger)^*b]^\dagger aa^\dagger(a^\dagger)^*bb^*$ are self-adjoint too. Hence, we see that $[(a^*a)^\dagger bb^*]^\dagger = (b^\dagger)^*[(a^\dagger)^*b]^\dagger a$.

Now we check that $[(a^\dagger)^*b]^\dagger = b^*(b^\dagger)^*[(a^\dagger)^*b]^\dagger aa^\dagger = b^\dagger b[(a^\dagger)^*b]^\dagger aa^\dagger$:

$$(a^\dagger)^*bb^\dagger b[(a^\dagger)^*b]^\dagger aa^\dagger(a^\dagger)^*b = (a^\dagger)^*b[(a^\dagger)^*b]^\dagger(a^\dagger)^*b = (a^\dagger)^*b,$$

$$b^\dagger b[(a^\dagger)^*b]^\dagger aa^\dagger(a^\dagger)^*bb^\dagger b[(a^\dagger)^*b]^\dagger aa^\dagger = b^\dagger b[(a^\dagger)^*b]^\dagger(a^\dagger)^*b[(a^\dagger)^*b]^\dagger aa^\dagger = b^\dagger b[(a^\dagger)^*b]^\dagger aa^\dagger,$$

$$\begin{aligned}((a^\dagger)^*bb^\dagger b[(a^\dagger)^*b]^\dagger aa^\dagger)^* &= aa^\dagger(a^\dagger)^*b[(a^\dagger)^*b]^\dagger = (a^\dagger)^*b[(a^\dagger)^*b]^\dagger \\ &= ((a^\dagger)^*b[(a^\dagger)^*b]^\dagger)^* = (aa^\dagger(a^\dagger)^*b[(a^\dagger)^*b]^\dagger)^* \\ &= (a^\dagger)^*b[(a^\dagger)^*b]^\dagger aa^\dagger = (a^\dagger)^*bb^\dagger b[(a^\dagger)^*b]^\dagger aa^\dagger,\end{aligned}$$

$$\begin{aligned}(b^\dagger b[(a^\dagger)^*b]^\dagger aa^\dagger(a^\dagger)^*b)^* &= [(a^\dagger)^*b]^\dagger(a^\dagger)^*bb^\dagger b = [(a^\dagger)^*b]^\dagger(a^\dagger)^*b \\ &= ([(a^\dagger)^*b]^\dagger(a^\dagger)^*b)^* = ([(a^\dagger)^*b]^\dagger(a^\dagger)^*bb^\dagger b)^* \\ &= b^\dagger b[(a^\dagger)^*b]^\dagger(a^\dagger)^*b = b^\dagger b[(a^\dagger)^*b]^\dagger aa^\dagger(a^\dagger)^*b.\end{aligned}$$

Finally, by the equality $[(a^*a)^\dagger bb^*]^\dagger = (b^\dagger)^*[(a^\dagger)^*b]^\dagger a$, we have

$$[(a^\dagger)^*b]^\dagger = b^*(b^\dagger)^*[(a^\dagger)^*b]^\dagger aa^\dagger = b^*[(a^*a)^\dagger bb^*]^\dagger a^\dagger.$$

Thus, the condition (b2) is satisfied.

(b2) $\Rightarrow$ (b1): To prove $a^\dagger abb^\dagger \in \mathcal{R}^\dagger$ and $(a^\dagger abb^\dagger)^\dagger = b[(a^\dagger)^*b]^\dagger(a^\dagger)^*$, notice that

$$\begin{aligned} a^\dagger abb^\dagger b[(a^\dagger)^*b]^\dagger(a^\dagger)^*a^\dagger abb^\dagger &= a^*((a^\dagger)^*b[(a^\dagger)^*b]^\dagger(a^\dagger)^*b)b^\dagger = a^*(a^\dagger)^*bb^\dagger \\ &= a^\dagger abb^\dagger, \end{aligned} \tag{5}$$

$$\begin{aligned} b[(a^\dagger)^*b]^\dagger(a^\dagger)^*a^\dagger abb^\dagger b[(a^\dagger)^*b]^\dagger(a^\dagger)^* &= b[(a^\dagger)^*b]^\dagger(a^\dagger)^*b[(a^\dagger)^*b]^\dagger(a^\dagger)^* \\ &= b[(a^\dagger)^*b]^\dagger(a^\dagger)^*, \end{aligned} \tag{6}$$

i.e. $b[(a^\dagger)^*b]^\dagger(a^\dagger)^* \in (a^\dagger abb^\dagger)\{1,2\}$. Using the assumption $[(a^\dagger)^*b]^\dagger = b^*[(a^*a)^\dagger bb^*]^\dagger a^\dagger$, we get

$$\begin{aligned} a^\dagger abb^\dagger b[(a^\dagger)^*b]^\dagger(a^\dagger)^* &= a^*(a^\dagger)^*b[(a^\dagger)^*b]^\dagger(a^\dagger)^* = (a^\dagger(a^\dagger)^*b[(a^\dagger)^*b]^\dagger a)^* \\ &= ((a^*a)^\dagger bb^*[(a^*a)^\dagger bb^*]^\dagger a^\dagger a)^* \\ &= a^\dagger a(a^*a)^\dagger bb^*[(a^*a)^\dagger bb^*]^\dagger = a^\dagger(a^\dagger)^*bb^*[(a^*a)^\dagger bb^*]^\dagger \\ &= (a^*a)^\dagger bb^*[(a^*a)^\dagger bb^*]^\dagger \end{aligned}$$

and

$$\begin{aligned} b[(a^\dagger)^*b]^\dagger(a^\dagger)^*a^\dagger abb^\dagger &= b[(a^\dagger)^*b]^\dagger(a^\dagger)^*bb^\dagger = ((b^\dagger)^*[(a^\dagger)^*b]^\dagger(a^\dagger)^*bb^*)^* \\ &= ((b^\dagger)^*b^*[(a^*a)^\dagger bb^*]^\dagger a^\dagger(a^\dagger)^*bb^*)^* \\ &= ((b^\dagger)^*b^*[(a^*a)^\dagger bb^*]^\dagger(a^*a)^\dagger bb^*)^* \\ &= [(a^*a)^\dagger bb^*]^\dagger(a^*a)^\dagger bb^*bb^\dagger \\ &= [(a^*a)^\dagger bb^*]^\dagger(a^*a)^\dagger bb^*, \end{aligned}$$

i.e. $a^\dagger abb^\dagger b[(a^\dagger)^*b]^\dagger(a^\dagger)^*$ and $b[(a^\dagger)^*b]^\dagger(a^\dagger)^*a^\dagger abb^\dagger$ are self-adjoint elements. Consequently, $a^\dagger abb^\dagger \in \mathcal{R}^\dagger$ and $(a^\dagger abb^\dagger)^\dagger = b[(a^\dagger)^*b]^\dagger(a^\dagger)^*$. Then, we will show that $[(a^\dagger)^*b]^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^*$. The equalities

$$(a^\dagger)^*bb^\dagger(a^\dagger abb^\dagger)^\dagger a^*(a^\dagger)^*b = (a^\dagger)^*(a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger)b = (a^\dagger)^*a^\dagger abb^\dagger b = (a^\dagger)^*b,$$

$$b^\dagger(a^\dagger abb^\dagger)^\dagger a^*(a^\dagger)^*bb^\dagger(a^\dagger abb^\dagger)^\dagger a^* = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^* = b^\dagger(a^\dagger abb^\dagger)^\dagger a^*,$$

yield $b^\dagger(a^\dagger abb^\dagger)^\dagger a^* \in [(a^\dagger)^*b]\{1,2\}$. By $(a^\dagger abb^\dagger)^\dagger = b[(a^\dagger)^*b]^\dagger(a^\dagger)^*$, we get that

$$\begin{aligned} (a^\dagger)^*bb^\dagger(a^\dagger abb^\dagger)^\dagger a^* &= (a^\dagger)^*bb^\dagger b[(a^\dagger)^*b]^\dagger(a^\dagger)^*a^* = (aa^\dagger(a^\dagger)^*b[(a^\dagger)^*b]^\dagger)^* \\ &= ((a^\dagger)^*b[(a^\dagger)^*b]^\dagger)^* = (a^\dagger)^*b[(a^\dagger)^*b]^\dagger, \end{aligned}$$

$$\begin{aligned} b^\dagger(a^\dagger abb^\dagger)^\dagger a^*(a^\dagger)^*b &= b^\dagger b[(a^\dagger)^*b]^\dagger(a^\dagger)^*a^*(a^\dagger)^*b = b^\dagger b[(a^\dagger)^*b]^\dagger(a^\dagger)^*b \\ &= ([(a^\dagger)^*b]^\dagger(a^\dagger)^*bb^\dagger b)^* = ([(a^\dagger)^*b]^\dagger(a^\dagger)^*b)^* = [(a^\dagger)^*b]^\dagger(a^\dagger)^*b, \end{aligned}$$

implying $b^\dagger(a^\dagger abb^\dagger)^\dagger a^* \in [(a^\dagger)^* b]\{3,4\}$. Hence, the statement (b1) holds.

(c1) ⇒ (c2): By definition, we check that $a^*a(bb^*)^\dagger = a^*a(b^\dagger)^*b^\dagger \in \mathcal{R}^\dagger$ and $[a^*a(bb^*)^\dagger]^\dagger = b[a(b^\dagger)^*]^\dagger(a^\dagger)^*$. From

$$a^*a(b^\dagger)^*b^\dagger b[a(b^\dagger)^*]^\dagger(a^\dagger)^*a^*a(b^\dagger)^*b^\dagger = a^*(a(b^\dagger)^*[a(b^\dagger)^*]^\dagger a(b^\dagger)^*)b^\dagger = a^*a(b^\dagger)^*b^\dagger \tag{7}$$

and

$$\begin{aligned} b[a(b^\dagger)^*]^\dagger(a^\dagger)^*a^*a(b^\dagger)^*b^\dagger b[a(b^\dagger)^*]^\dagger(a^\dagger)^* &= b[a(b^\dagger)^*]^\dagger a(b^\dagger)^*[a(b^\dagger)^*]^\dagger(a^\dagger)^* \\ &= b[a(b^\dagger)^*]^\dagger(a^\dagger)^* \end{aligned} \tag{8}$$

we deduce that $b[a(b^\dagger)^*]^\dagger(a^\dagger)^* \in (a^*a(bb^*)^\dagger)\{1,2\}$. The condition $[a(b^\dagger)^*]^\dagger = b^*(a^\dagger abb^\dagger)^\dagger a^\dagger$ gives

$$\begin{aligned} a^*a(b^\dagger)^*b^\dagger b[a(b^\dagger)^*]^\dagger(a^\dagger)^* &= a^*a(b^\dagger)^*[a(b^\dagger)^*]^\dagger(a^\dagger)^* = (a^\dagger a(b^\dagger)^*[a(b^\dagger)^*]^\dagger a)^* \\ &= (a^\dagger a(b^\dagger)^*b^*(a^\dagger abb^\dagger)^\dagger a^\dagger a)^* = (a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger a)^* \\ &= a^\dagger aa^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger = a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger \text{ is self-adjoint} \end{aligned}$$

and

$$\begin{aligned} b[a(b^\dagger)^*]^\dagger(a^\dagger)^*a^*a(b^\dagger)^*b^\dagger &= b[a(b^\dagger)^*]^\dagger a(b^\dagger)^*b^\dagger = ((b^\dagger)^*[a(b^\dagger)^*]^\dagger a(b^\dagger)^*b^*)^* \\ &= ((b^\dagger)^*b^*(a^\dagger abb^\dagger)^\dagger a^\dagger a(b^\dagger)^*b^*)^* = (bb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger)^* \\ &= (a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger bb^\dagger = (a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger \text{ is self-adjoint.} \end{aligned}$$

Thus, $a^*a(bb^*)^\dagger \in \mathcal{R}^\dagger$ and $[a^*a(bb^*)^\dagger]^\dagger = b[a(b^\dagger)^*]^\dagger(a^\dagger)^*$. To obtain the equality $[a(b^\dagger)^*]^\dagger = b^\dagger[a^*a(bb^*)^\dagger]^\dagger a^*$ it is enough to prove that $[a(b^\dagger)^*]^\dagger = b^\dagger b[a(b^\dagger)^*]^\dagger(a^\dagger)^*a^* = b^\dagger b[a(b^\dagger)^*]^\dagger aa^\dagger$. Since

$$a(b^\dagger)^*b^\dagger b[a(b^\dagger)^*]^\dagger aa^\dagger a(b^\dagger)^* = a(b^\dagger)^*[a(b^\dagger)^*]^\dagger a(b^\dagger)^* = a(b^\dagger)^*,$$

$$b^\dagger b[a(b^\dagger)^*]^\dagger aa^\dagger a(b^\dagger)^*b^\dagger b[a(b^\dagger)^*]^\dagger aa^\dagger = b^\dagger b[a(b^\dagger)^*]^\dagger a(b^\dagger)^*[a(b^\dagger)^*]^\dagger aa^\dagger = b^\dagger b[a(b^\dagger)^*]^\dagger aa^\dagger,$$

$$\begin{aligned} (a(b^\dagger)^*b^\dagger b[a(b^\dagger)^*]^\dagger aa^\dagger)^* &= (a(b^\dagger)^*[a(b^\dagger)^*]^\dagger aa^\dagger)^* = aa^\dagger a(b^\dagger)^*[a(b^\dagger)^*]^\dagger \\ &= a(b^\dagger)^*[a(b^\dagger)^*]^\dagger \text{ is self-adjoint,} \end{aligned}$$

$$\begin{aligned} (b^\dagger b[a(b^\dagger)^*]^\dagger aa^\dagger a(b^\dagger)^*)^* &= (b^\dagger b[a(b^\dagger)^*]^\dagger a(b^\dagger)^*)^* = [a(b^\dagger)^*]^\dagger a(b^\dagger)^*b^\dagger b \\ &= [a(b^\dagger)^*]^\dagger a(b^\dagger)^* \text{ is self-adjoint,} \end{aligned}$$

then $[a(b^\dagger)^*]^\dagger = b^\dagger b[a(b^\dagger)^*]^\dagger aa^\dagger = b^\dagger[a^*a(bb^*)^\dagger]^\dagger a^*$ and (c2) is satisfied.

(c2) ⇒ (c1): First we will prove that $a^\dagger abb^\dagger \in \mathcal{R}^\dagger$ and $(a^\dagger abb^\dagger)^\dagger = (b^\dagger)^*[a(b^\dagger)^*]^\dagger a$. The equalities

$$a^\dagger abb^\dagger(b^\dagger)^*[a(b^\dagger)^*]^\dagger aa^\dagger abb^\dagger = a^\dagger(a(b^\dagger)^*[a(b^\dagger)^*]^\dagger a(b^\dagger)^*)b^* = a^\dagger a(b^\dagger)^*b^* = a^\dagger abb^\dagger,$$

$$(b^\dagger)^*[a(b^\dagger)^*]^\dagger aa^\dagger abb^\dagger(b^\dagger)^*[a(b^\dagger)^*]^\dagger a = (b^\dagger)^*[a(b^\dagger)^*]^\dagger a(b^\dagger)^*[a(b^\dagger)^*]^\dagger a = (b^\dagger)^*[a(b^\dagger)^*]^\dagger a,$$

imply that $(b^\dagger)^*[a(b^\dagger)^*]^\dagger a \in (a^\dagger abb^\dagger)\{1,2\}$. Using the hypothesis $[a(b^\dagger)^*]^\dagger = b^\dagger[a^*a(bb^*)^\dagger]^\dagger a^*$, we get that

$$\begin{aligned} a^\dagger abb^\dagger(b^\dagger)^*[a(b^\dagger)^*]^\dagger a &= a^\dagger a(b^\dagger)^*[a(b^\dagger)^*]^\dagger a = (a^*a(b^\dagger)^*[a(b^\dagger)^*]^\dagger(a^\dagger)^*)^* \\ &= (a^*a(b^\dagger)^*b^\dagger[a^*a(bb^*)^\dagger]^\dagger a^*(a^\dagger)^*)^* = (a^*a(bb^*)^\dagger[a^*a(bb^*)^\dagger]^\dagger a^*(a^\dagger)^*)^* \\ &= a^\dagger aa^*a(bb^*)^\dagger[a^*a(bb^*)^\dagger]^\dagger = a^*a(bb^*)^\dagger[a^*a(bb^*)^\dagger]^\dagger \text{ is self} - \text{adjoint} \end{aligned}$$

and

$$\begin{aligned} (b^\dagger)^*[a(b^\dagger)^*]^\dagger aa^\dagger abb^\dagger &= (b^\dagger)^*[a(b^\dagger)^*]^\dagger a(b^\dagger)^*b^* = (b[a(b^\dagger)^*]^\dagger a(b^\dagger)^*b^\dagger)^* \\ &= (bb^\dagger[a^*a(bb^*)^\dagger]^\dagger a^*a(b^\dagger)^*b^\dagger)^* = (bb^\dagger[a^*a(bb^*)^\dagger]^\dagger a^*a(bb^*)^\dagger)^* \\ &= [a^*a(bb^*)^\dagger]^\dagger a^*a(bb^*)^\dagger bb^\dagger = [a^*a(bb^*)^\dagger]^\dagger a^*a(b^\dagger)^*b^\dagger \\ &= [a^*a(bb^*)^\dagger]^\dagger a^*a(bb^*)^\dagger \text{ is self} - \text{adjoint.} \end{aligned}$$

Hence, we conclude that $(a^\dagger abb^\dagger)^\dagger = (b^\dagger)^*[a(b^\dagger)^*]^\dagger a$. Now in order to show that the equality $[a(b^\dagger)^*]^\dagger = b^*(a^\dagger abb^\dagger)^\dagger a^\dagger$ holds, we prove that $[a(b^\dagger)^*]^\dagger = b^*(b^\dagger)^*[a(b^\dagger)^*]^\dagger aa^\dagger = b^\dagger b[a(b^\dagger)^*]^\dagger aa^\dagger$. Indeed, by definition and

$$a(b^\dagger)^*b^\dagger b[a(b^\dagger)^*]^\dagger aa^\dagger a(b^\dagger)^* = a(b^\dagger)^*[a(b^\dagger)^*]^\dagger a(b^\dagger)^* = a(b^\dagger)^*,$$

$$b^\dagger b[a(b^\dagger)^*]^\dagger aa^\dagger a(b^\dagger)^*b^\dagger b[a(b^\dagger)^*]^\dagger aa^\dagger = b^\dagger b[a(b^\dagger)^*]^\dagger a(b^\dagger)^*[a(b^\dagger)^*]^\dagger aa^\dagger = b^\dagger b[a(b^\dagger)^*]^\dagger aa^\dagger,$$

$$(a(b^\dagger)^*b^\dagger b[a(b^\dagger)^*]^\dagger aa^\dagger)^* = (a(b^\dagger)^*[a(b^\dagger)^*]^\dagger aa^\dagger)^* = aa^\dagger a(b^\dagger)^*[a(b^\dagger)^*]^\dagger = a(b^\dagger)^*[a(b^\dagger)^*]^\dagger \text{ is self} - \text{adjoint,}$$

$$(b^\dagger b[a(b^\dagger)^*]^\dagger aa^\dagger a(b^\dagger)^*)^* = (b^\dagger b[a(b^\dagger)^*]^\dagger a(b^\dagger)^*)^* = [a(b^\dagger)^*]^\dagger a(b^\dagger)^*b^\dagger b = [a(b^\dagger)^*]^\dagger a(b^\dagger)^* \text{ is self} - \text{adjoint.}$$

we have $[a(b^\dagger)^*]^\dagger = b^\dagger b[a(b^\dagger)^*]^\dagger aa^\dagger = b^*(a^\dagger abb^\dagger)^\dagger a^\dagger$. So, the condition (c1) is satisfied.

(d1) ⇒ (d2): Let us check that $(bb^*)^\dagger(a^*a)^\dagger = (b^\dagger)^*b^\dagger a^\dagger(a^\dagger)^* \in \mathcal{R}^\dagger$ and $[(b^\dagger)^*b^\dagger a^\dagger(a^\dagger)^*]^\dagger = a^*(b^\dagger a^\dagger)^\dagger b^*$. By

$$\begin{aligned} (b^\dagger)^*b^\dagger a^\dagger(a^\dagger)^*a^*(b^\dagger a^\dagger)^\dagger b^*(b^\dagger)^*b^\dagger a^\dagger(a^\dagger)^* &= (b^\dagger)^*b^\dagger a^\dagger(b^\dagger a^\dagger)^\dagger b^\dagger a^\dagger(a^\dagger)^* \\ &= (b^\dagger)^*b^\dagger a^\dagger(a^\dagger)^*, \end{aligned} \tag{9}$$

$$\begin{aligned} a^*(b^\dagger a^\dagger)^\dagger b^*(b^\dagger)^*b^\dagger a^\dagger(a^\dagger)^*a^*(b^\dagger a^\dagger)^\dagger b^* &= a^*(b^\dagger a^\dagger)^\dagger b^\dagger a^\dagger(b^\dagger a^\dagger)^\dagger b^* \\ &= a^*(b^\dagger a^\dagger)^\dagger b^*, \end{aligned} \tag{10}$$

obviously, $a^*(b^\dagger a^\dagger)^\dagger b^* \in [(b^\dagger)^*b^\dagger a^\dagger(a^\dagger)^*]\{1,2\}$. Further, from the condition $(b^\dagger a^\dagger)^\dagger = a(bb^\dagger a^\dagger a)^\dagger b$, we get

$$\begin{aligned} (b^\dagger)^*b^\dagger a^\dagger(a^\dagger)^*a^*(b^\dagger a^\dagger)^\dagger b^* &= (b^\dagger)^*b^\dagger a^\dagger(b^\dagger a^\dagger)^\dagger b^* = (bb^\dagger a^\dagger(b^\dagger a^\dagger)^\dagger b^\dagger)^* \\ &= (bb^\dagger a^\dagger a(bb^\dagger a^\dagger a)^\dagger bb^\dagger)^* = bb^\dagger bb^\dagger a^\dagger a(bb^\dagger a^\dagger a)^\dagger \\ &= bb^\dagger a^\dagger a(bb^\dagger a^\dagger a)^\dagger \text{ is self} - \text{adjoint} \end{aligned}$$

and

$$\begin{aligned} a^*(b^\dagger a^\dagger)^\dagger b^*(b^\dagger)^*b^\dagger a^\dagger(a^\dagger)^* &= a^*(b^\dagger a^\dagger)^\dagger b^\dagger a^\dagger(a^\dagger)^* = (a^\dagger(b^\dagger a^\dagger)^\dagger b^\dagger a^\dagger a)^* \\ &= (a^\dagger a(bb^\dagger a^\dagger a)^\dagger bb^\dagger a^\dagger a)^* = (bb^\dagger a^\dagger a)^\dagger bb^\dagger a^\dagger aa^\dagger a \\ &= (bb^\dagger a^\dagger a)^\dagger bb^\dagger a^\dagger a \text{ is self} - \text{adjoint,} \end{aligned}$$

i.e. $a^*(b^\dagger a^\dagger)^\dagger b^* \in [(b^\dagger)^* b^\dagger a^\dagger (a^\dagger)^*]\{3,4\}$. Thus, $[(bb^*)^\dagger (a^*a)^\dagger]^\dagger = a^*(b^\dagger a^\dagger)^\dagger b^*$. Then, a direct computation shows that $(b^\dagger a^\dagger)^\dagger = (a^\dagger)^* a^*(b^\dagger a^\dagger)^\dagger b^*(b^\dagger)^* = aa^\dagger (b^\dagger a^\dagger)^\dagger b^\dagger b$:

$$b^\dagger a^\dagger aa^\dagger (b^\dagger a^\dagger)^\dagger b^\dagger bb^\dagger a^\dagger = b^\dagger a^\dagger (b^\dagger a^\dagger)^\dagger b^\dagger a^\dagger = b^\dagger a^\dagger,$$

$$aa^\dagger (b^\dagger a^\dagger)^\dagger b^\dagger bb^\dagger a^\dagger aa^\dagger (b^\dagger a^\dagger)^\dagger b^\dagger b = aa^\dagger (b^\dagger a^\dagger)^\dagger b^\dagger a^\dagger (b^\dagger a^\dagger)^\dagger b^\dagger b = aa^\dagger (b^\dagger a^\dagger)^\dagger b^\dagger b,$$

$$\begin{aligned}(b^\dagger a^\dagger aa^\dagger (b^\dagger a^\dagger)^\dagger b^\dagger b)^* &= (b^\dagger a^\dagger (b^\dagger a^\dagger)^\dagger b^\dagger b)^* = b^\dagger bb^\dagger a^\dagger (b^\dagger a^\dagger)^\dagger \\ &= b^\dagger a^\dagger (b^\dagger a^\dagger)^\dagger \text{ is self-adjoint,}\end{aligned}$$

$$\begin{aligned}(aa^\dagger (b^\dagger a^\dagger)^\dagger b^\dagger bb^\dagger a^\dagger)^* &= (aa^\dagger (b^\dagger a^\dagger)^\dagger b^\dagger a^\dagger)^* = (b^\dagger a^\dagger)^\dagger b^\dagger a^\dagger aa^\dagger \\ &= (b^\dagger a^\dagger)^\dagger b^\dagger a^\dagger \text{ is self-adjoint.}\end{aligned}$$

Therefore, $(b^\dagger a^\dagger)^\dagger = (a^\dagger)^* a^*(b^\dagger a^\dagger)^\dagger b^*(b^\dagger)^* = (a^\dagger)^*[(bb^*)^\dagger (a^*a)^\dagger]^\dagger (b^\dagger)^*$, by $[(bb^*)^\dagger (a^*a)^\dagger]^\dagger = a^*(b^\dagger a^\dagger)^\dagger b^*$.

(d2) ⇒ (d1): To prove $bb^\dagger a^\dagger a \in \mathcal{R}^\dagger$ and $(bb^\dagger a^\dagger a)^\dagger = a^\dagger (b^\dagger a^\dagger)^\dagger b^\dagger$, first we have $a^\dagger (b^\dagger a^\dagger)^\dagger b^\dagger \in (bb^\dagger a^\dagger a)\{1,2\}$, from

$$bb^\dagger a^\dagger aa^\dagger (b^\dagger a^\dagger)^\dagger b^\dagger bb^\dagger a^\dagger a = b(b^\dagger a^\dagger (b^\dagger a^\dagger)^\dagger b^\dagger a^\dagger)a = bb^\dagger a^\dagger a,$$

$$a^\dagger (b^\dagger a^\dagger)^\dagger b^\dagger bb^\dagger a^\dagger aa^\dagger (b^\dagger a^\dagger)^\dagger b^\dagger = a^\dagger (b^\dagger a^\dagger)^\dagger b^\dagger a^\dagger (b^\dagger a^\dagger)^\dagger b^\dagger = a^\dagger (b^\dagger a^\dagger)^\dagger b^\dagger.$$

We use the hypothesis $(b^\dagger a^\dagger)^\dagger = (a^\dagger)^*[(bb^*)^\dagger (a^*a)^\dagger]^\dagger (b^\dagger)^*$ to obtain that $a^\dagger (b^\dagger a^\dagger)^\dagger b^\dagger \in (bb^\dagger a^\dagger a)\{3,4\}$ in the following way:

$$\begin{aligned}bb^\dagger a^\dagger aa^\dagger (b^\dagger a^\dagger)^\dagger b^\dagger &= bb^\dagger a^\dagger (b^\dagger a^\dagger)^\dagger b^\dagger = ((b^\dagger)^* b^\dagger a^\dagger (b^\dagger a^\dagger)^\dagger b^*)^* = ((b^\dagger)^* b^\dagger a^\dagger (a^\dagger)^*[(bb^*)^\dagger (a^*a)^\dagger]^\dagger (b^\dagger)^* b^*)^* \\ &= ((bb^*)^\dagger (a^*a)^\dagger [(bb^*)^\dagger (a^*a)^\dagger]^\dagger (b^\dagger)^* b^*)^* = bb^\dagger (bb^*)^\dagger (a^*a)^\dagger [(bb^*)^\dagger (a^*a)^\dagger]^\dagger \\ &= bb^\dagger (b^\dagger)^* b^\dagger (a^*a)^\dagger [(bb^*)^\dagger (a^*a)^\dagger]^\dagger = (bb^*)^\dagger (a^*a)^\dagger [(bb^*)^\dagger (a^*a)^\dagger]^\dagger \text{ is self-adjoint,}\end{aligned}$$

$$\begin{aligned}a^\dagger (b^\dagger a^\dagger)^\dagger b^\dagger bb^\dagger a^\dagger a &= a^\dagger (b^\dagger a^\dagger)^\dagger b^\dagger a^\dagger a = (a^*(b^\dagger a^\dagger)^\dagger b^\dagger a^\dagger (a^\dagger)^*)^* = (a^*(a^\dagger)^*[(bb^*)^\dagger (a^*a)^\dagger]^\dagger (b^\dagger)^* b^\dagger a^\dagger (a^\dagger)^*)^* \\ &= (a^*(a^\dagger)^*[(bb^*)^\dagger (a^*a)^\dagger]^\dagger (bb^*)^\dagger (a^*a)^\dagger)^* = [(bb^*)^\dagger (a^*a)^\dagger]^\dagger (bb^*)^\dagger (a^*a)^\dagger a^\dagger a \\ &= [(bb^*)^\dagger (a^*a)^\dagger]^\dagger (bb^*)^\dagger a^\dagger (a^\dagger)^* a^\dagger a = [(bb^*)^\dagger (a^*a)^\dagger]^\dagger (bb^*)^\dagger (a^*a)^\dagger \text{ is self-adjoint.}\end{aligned}$$

So, $(bb^\dagger a^\dagger a)^\dagger = a^\dagger (b^\dagger a^\dagger)^\dagger b^\dagger$ and then to obtain $(b^\dagger a^\dagger)^\dagger = a(bb^\dagger a^\dagger a)^\dagger b$ it is enough to check that $(b^\dagger a^\dagger)^\dagger = aa^\dagger (b^\dagger a^\dagger)^\dagger b^\dagger b$:

$$b^\dagger a^\dagger aa^\dagger (b^\dagger a^\dagger)^\dagger b^\dagger bb^\dagger a^\dagger = b^\dagger a^\dagger (b^\dagger a^\dagger)^\dagger b^\dagger a^\dagger = b^\dagger a^\dagger,$$

$$aa^\dagger (b^\dagger a^\dagger)^\dagger b^\dagger bb^\dagger a^\dagger aa^\dagger (b^\dagger a^\dagger)^\dagger b^\dagger b = aa^\dagger (b^\dagger a^\dagger)^\dagger b^\dagger a^\dagger (b^\dagger a^\dagger)^\dagger b^\dagger b = aa^\dagger (b^\dagger a^\dagger)^\dagger b^\dagger b,$$

$$\begin{aligned}(b^\dagger a^\dagger aa^\dagger (b^\dagger a^\dagger)^\dagger b^\dagger b)^* &= (b^\dagger a^\dagger (b^\dagger a^\dagger)^\dagger b^\dagger b)^* = b^\dagger bb^\dagger a^\dagger (b^\dagger a^\dagger)^\dagger \\ &= b^\dagger a^\dagger (b^\dagger a^\dagger)^\dagger \text{ is self-adjoint,}\end{aligned}$$

$$\begin{aligned}(aa^\dagger (b^\dagger a^\dagger)^\dagger b^\dagger bb^\dagger a^\dagger)^* &= (aa^\dagger (b^\dagger a^\dagger)^\dagger b^\dagger a^\dagger)^* = (b^\dagger a^\dagger)^\dagger b^\dagger a^\dagger aa^\dagger \\ &= (b^\dagger a^\dagger)^\dagger b^\dagger a^\dagger \text{ is self-adjoint.}\end{aligned}$$

Thus, the condition (d1) is satisfied.

(a1) ⇒ (e1): This implication follows from Theorem 1.5.

(e1) ⇒ (a1): We will verify that $a^\dagger abb^\dagger \in \mathcal{R}^\dagger$ and $(a^\dagger abb^\dagger)^\dagger = b(a^\dagger ab)^\dagger$. Obviously,

$$a^\dagger abb^\dagger b(a^\dagger ab)^\dagger a^\dagger abb^\dagger = (a^\dagger ab(a^\dagger ab)^\dagger a^\dagger ab)b^\dagger = a^\dagger abb^\dagger, \tag{11}$$

$$b(a^\dagger ab)^\dagger a^\dagger abb^\dagger b(a^\dagger ab)^\dagger = b(a^\dagger ab)^\dagger a^\dagger ab(a^\dagger ab)^\dagger = b(a^\dagger ab)^\dagger, \tag{12}$$

$$a^\dagger abb^\dagger b(a^\dagger ab)^\dagger = a^\dagger ab(a^\dagger ab)^\dagger \text{ is self} - \text{adjoint}. \tag{13}$$

From $(a^\dagger ab)^\dagger a^\dagger = b^\dagger(abb^\dagger)^\dagger$, we have

$$b(a^\dagger ab)^\dagger a^\dagger abb^\dagger = bb^\dagger(abb^\dagger)^\dagger abb^\dagger = ((abb^\dagger)^\dagger abb^\dagger bb^\dagger)^* = (abb^\dagger)^\dagger abb^\dagger,$$

which implies that element $b(a^\dagger ab)^\dagger a^\dagger abb^\dagger$ is self-adjoint. Thus, the conditions $a^\dagger abb^\dagger \in \mathcal{R}^\dagger$ and $(a^\dagger abb^\dagger)^\dagger = b(a^\dagger ab)^\dagger$ hold. By this equality and (e1), we obtain

$$b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger = b^\dagger b(a^\dagger ab)^\dagger a^\dagger = b^\dagger bb^\dagger(abb^\dagger)^\dagger = b^\dagger(abb^\dagger)^\dagger. \tag{14}$$

From

$$abb^\dagger(abb^\dagger)^\dagger ab = (abb^\dagger(abb^\dagger)^\dagger abb^\dagger)b = abb^\dagger b = ab,$$

$$b^\dagger(abb^\dagger)^\dagger abb^\dagger(abb^\dagger)^\dagger = b^\dagger(abb^\dagger)^\dagger,$$

we conclude that $b^\dagger(abb^\dagger)^\dagger \in (ab)\{1,2\}$. Next, $abb^\dagger(abb^\dagger)^\dagger$ is self-adjoint and, by (e1), $b^\dagger(abb^\dagger)^\dagger ab = (a^\dagger ab)^\dagger a^\dagger ab$ is self-adjoint too. Consequently, $ab \in \mathcal{R}^\dagger$ and $(ab)^\dagger = b^\dagger(abb^\dagger)^\dagger$. Then, by (14), we observe that $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$. Hence, the statement (a1) is satisfied. Notice that from (e1) follows $(ab)^\dagger = b^\dagger(abb^\dagger)^\dagger = (a^\dagger ab)^\dagger a^\dagger$.

(b1) ⇒ (e2): Let us remark that $b^\dagger(a^\dagger abb^\dagger)^\dagger \in (a^\dagger ab)\{1,2,3\}$ follows from

$$a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger ab = (a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger)b = a^\dagger abb^\dagger b = a^\dagger ab,$$

$$b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger,$$

$$a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger \text{ is self} - \text{adjoint}.$$

Similarly, $(a^\dagger abb^\dagger)^\dagger a^* \in [(a^\dagger)^* bb^\dagger]\{1,2,4\}$ follows from

$$(a^\dagger)^* bb^\dagger(a^\dagger abb^\dagger)^\dagger a^*(a^\dagger)^* bb^\dagger = (a^\dagger)^*(a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger) = (a^\dagger)^* a^\dagger abb^\dagger = (a^\dagger)^* bb^\dagger,$$

$$(a^\dagger abb^\dagger)^\dagger a^*(a^\dagger)^* bb^\dagger(a^\dagger abb^\dagger)^\dagger a^* = (a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^* = (a^\dagger abb^\dagger)^\dagger a^*,$$

$$(a^\dagger abb^\dagger)^\dagger a^*(a^\dagger)^* bb^\dagger = (a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger \text{ is self} - \text{adjoint}.$$

The assumption $[(a^\dagger)^* b]^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^*$ gives that

$$b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger ab = b^\dagger(a^\dagger abb^\dagger)^\dagger a^*(a^\dagger)^* b \text{ is self} - \text{adjoint},$$

$$(a^\dagger)^*bb^\dagger(a^\dagger abb^\dagger)^\dagger a^* \text{ is self}-\text{adjoint},$$

i.e. $b^\dagger(a^\dagger abb^\dagger)^\dagger \in (a^\dagger ab)\{4\}$ and $(a^\dagger abb^\dagger)^\dagger a^* \in [(a^\dagger)^*bb^\dagger]\{3\}$. Therefore, $a^\dagger ab, (a^\dagger)^*bb^\dagger \in \mathcal{R}^\dagger$, $(a^\dagger ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger$ and $[(a^\dagger)^*bb^\dagger]^\dagger = (a^\dagger abb^\dagger)^\dagger a^*$. Now, $(a^\dagger ab)^\dagger a^* = b^\dagger(a^\dagger abb^\dagger)^\dagger a^*(= [(a^\dagger)^*b]^\dagger) = b^\dagger[(a^\dagger)^*bb^\dagger]^\dagger$, i.e. the condition (e2) is satisfied.

(e2) ⇒ (b1): Notice that, by (11), (12) and (13), we have $b(a^\dagger ab)^\dagger \in (a^\dagger abb^\dagger)\{1,2,3\}$. The condition $(a^\dagger ab)^\dagger a^* = b^\dagger[(a^\dagger)^*bb^\dagger]^\dagger$ implies

$$\begin{aligned} b(a^\dagger ab)^\dagger a^\dagger abb^\dagger &= b(a^\dagger ab)^\dagger a^*(a^\dagger)^*bb^\dagger = bb^\dagger[(a^\dagger)^*bb^\dagger]^\dagger(a^\dagger)^*bb^\dagger \\ &= ([(a^\dagger)^*bb^\dagger]^\dagger(a^\dagger)^*bb^\dagger bb^\dagger)^* = ([(a^\dagger)^*bb^\dagger]^\dagger(a^\dagger)^*bb^\dagger)^* \\ &= [(a^\dagger)^*bb^\dagger]^\dagger(a^\dagger)^*bb^\dagger \text{ is self}-\text{adjoint}. \end{aligned}$$

So, $a^\dagger abb^\dagger \in \mathcal{R}^\dagger$ and $(a^\dagger abb^\dagger)^\dagger = b(a^\dagger ab)^\dagger$. Then

$$b^\dagger(a^\dagger abb^\dagger)^\dagger a^* = b^\dagger b(a^\dagger ab)^\dagger a^* = b^\dagger bb^\dagger[(a^\dagger)^*bb^\dagger]^\dagger = b^\dagger[(a^\dagger)^*bb^\dagger]^\dagger. \tag{15}$$

To get $[(a^\dagger)^*b]^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^*$, we will prove that $[(a^\dagger)^*b]^\dagger = b^\dagger[(a^\dagger)^*bb^\dagger]^\dagger$. Since

$$(a^\dagger)^*bb^\dagger[(a^\dagger)^*bb^\dagger]^\dagger(a^\dagger)^*b = ((a^\dagger)^*bb^\dagger[(a^\dagger)^*bb^\dagger]^\dagger(a^\dagger)^*bb^\dagger)b = (a^\dagger)^*bb^\dagger b = (a^\dagger)^*b,$$

$$b^\dagger[(a^\dagger)^*bb^\dagger]^\dagger(a^\dagger)^*bb^\dagger[(a^\dagger)^*bb^\dagger]^\dagger = b^\dagger[(a^\dagger)^*bb^\dagger]^\dagger,$$

$$(a^\dagger)^*bb^\dagger[(a^\dagger)^*bb^\dagger]^\dagger \text{ is self}-\text{adjoint},$$

we see that $b^\dagger[(a^\dagger)^*bb^\dagger]^\dagger \in [(a^\dagger)^*b]\{1,2,3\}$. Using (e2), we have

$$b^\dagger[(a^\dagger)^*bb^\dagger]^\dagger(a^\dagger)^*b = (a^\dagger ab)^\dagger a^*(a^\dagger)^*b = (a^\dagger ab)^\dagger a^\dagger ab,$$

i.e. $b^\dagger[(a^\dagger)^*bb^\dagger]^\dagger \in [(a^\dagger)^*b]\{4\}$. Thus, $(a^\dagger)^*b \in \mathcal{R}^\dagger$, $[(a^\dagger)^*b]^\dagger = b^\dagger[(a^\dagger)^*bb^\dagger]^\dagger$ and, by (15), $[(a^\dagger)^*b]^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^*$.

(c1) ⇒ (e3): By elementary computations, we obtain

$$a^\dagger a(b^\dagger)^*b^*(a^\dagger abb^\dagger)^\dagger a^\dagger a(b^\dagger)^* = (a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger)(b^\dagger)^* = a^\dagger abb^\dagger(b^\dagger)^* = a^\dagger a(b^\dagger)^*,$$

$$b^*(a^\dagger abb^\dagger)^\dagger a^\dagger a(b^\dagger)^*b^*(a^\dagger abb^\dagger)^\dagger = b^*(a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger = b^*(a^\dagger abb^\dagger)^\dagger,$$

$$a^\dagger a(b^\dagger)^*b^*(a^\dagger abb^\dagger)^\dagger = a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger \text{ is self}-\text{adjoint},$$

that is $b^*(a^\dagger abb^\dagger)^\dagger \in [a^\dagger a(b^\dagger)^*]\{1,2,3\}$. We easy check that $(a^\dagger abb^\dagger)^\dagger a^\dagger \in (abb^\dagger)\{1,2,4\}$:

$$abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger = a(a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger) = aa^\dagger abb^\dagger = abb^\dagger,$$

$$(a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger = (a^\dagger abb^\dagger)^\dagger a^\dagger,$$

$$(a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger \text{ is self}-\text{adjoint}.$$

The hypothesis $[a(b^\dagger)^*]^\dagger = b^*(a^\dagger abb^\dagger)^\dagger a^\dagger$ implies

$$b^*(a^\dagger abb^\dagger)^\dagger a^\dagger a(b^\dagger)^* \text{ is self}-\text{adjoint}$$

and

$$abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger = a(b^\dagger)^* b^*(a^\dagger abb^\dagger)^\dagger a^\dagger \text{ is self} - \text{adjoint}.$$

Consequently, the statements $a^\dagger a(b^\dagger)^*, abb^\dagger \in \mathcal{R}^\dagger$, $[a^\dagger a(b^\dagger)^*]^\dagger = b^*(a^\dagger abb^\dagger)^\dagger$ and $(abb^\dagger)^\dagger = (a^\dagger abb^\dagger)^\dagger a^\dagger$ hold. Finale, we get the equality in (e3), from $[a^\dagger a(b^\dagger)^*]^\dagger a^\dagger = b^*(a^\dagger abb^\dagger)^\dagger a^\dagger (= [a(b^\dagger)^*]^\dagger) = b^*(abb^\dagger)^\dagger$.

(e3) ⇒ (c1): First, we verify that $a^\dagger abb^\dagger \in \mathcal{R}^\dagger$ and $(a^\dagger abb^\dagger)^\dagger = (abb^\dagger)^\dagger a$. Indeed,

$$a^\dagger abb^\dagger(abb^\dagger)^\dagger aa^\dagger abb^\dagger = a^\dagger(abb^\dagger(abb^\dagger)^\dagger abb^\dagger) = a^\dagger abb^\dagger, \tag{16}$$

$$(abb^\dagger)^\dagger aa^\dagger abb^\dagger(abb^\dagger)^\dagger a = (abb^\dagger)^\dagger abb^\dagger(abb^\dagger)^\dagger a = (abb^\dagger)^\dagger a, \tag{17}$$

$$(abb^\dagger)^\dagger aa^\dagger abb^\dagger = (abb^\dagger)^\dagger abb^\dagger \text{ is self} - \text{adjoint}. \tag{18}$$

By the assumption $[a^\dagger a(b^\dagger)^*]^\dagger a^\dagger = b^*(abb^\dagger)^\dagger$, we have

$$\begin{aligned} a^\dagger abb^\dagger(abb^\dagger)^\dagger a &= a^\dagger a(b^\dagger)^* b^*(abb^\dagger)^\dagger a = a^\dagger a(b^\dagger)^*[a^\dagger a(b^\dagger)^*]^\dagger a^\dagger a \\ &= (a^\dagger aa^\dagger a(b^\dagger)^*[a^\dagger a(b^\dagger)^*]^\dagger)^* = (a^\dagger a(b^\dagger)^*[a^\dagger a(b^\dagger)^*]^\dagger)^* \\ &= a^\dagger a(b^\dagger)^*[a^\dagger a(b^\dagger)^*]^\dagger \text{ is self} - \text{adjoint}. \end{aligned} \tag{19}$$

Hence, by (16)-(19), $a^\dagger abb^\dagger \in \mathcal{R}^\dagger$ and $(a^\dagger abb^\dagger)^\dagger = (abb^\dagger)^\dagger a$. Further, we obtain $[a^\dagger a(b^\dagger)^*]^\dagger a^\dagger \in [a(b^\dagger)^*]\{1,2\}$ as a simple consequence of the equalities

$$a(b^\dagger)^*[a^\dagger a(b^\dagger)^*]^\dagger a^\dagger a(b^\dagger)^* = a(a^\dagger a(b^\dagger)^*[a^\dagger a(b^\dagger)^*]^\dagger a^\dagger a(b^\dagger)^*) = aa^\dagger a(b^\dagger)^* = a(b^\dagger)^*,$$

$$[a^\dagger a(b^\dagger)^*]^\dagger a^\dagger a(b^\dagger)^*[a^\dagger a(b^\dagger)^*]^\dagger a^\dagger = [a^\dagger a(b^\dagger)^*]^\dagger a^\dagger.$$

From (e3), we get

$$a(b^\dagger)^*[a^\dagger a(b^\dagger)^*]^\dagger a^\dagger = a(b^\dagger)^* b^*(abb^\dagger)^\dagger = abb^\dagger(abb^\dagger)^\dagger$$

which implies $[a^\dagger a(b^\dagger)^*]^\dagger a^\dagger \in [a(b^\dagger)^*]\{3\}$. Obviously, $[a^\dagger a(b^\dagger)^*]^\dagger a^\dagger a(b^\dagger)^*$ is self-adjoint and therefore, $a(b^\dagger)^* \in \mathcal{R}^\dagger$ and $[a(b^\dagger)^*]^\dagger = [a^\dagger a(b^\dagger)^*]^\dagger a^\dagger$. Now, by $(a^\dagger abb^\dagger)^\dagger = (abb^\dagger)^\dagger a$ and (e3),

$$b^*(a^\dagger abb^\dagger)^\dagger a^\dagger = b^*(abb^\dagger)^\dagger aa^\dagger = [a^\dagger a(b^\dagger)^*]^\dagger a^\dagger aa^\dagger = [a^\dagger a(b^\dagger)^*]^\dagger a^\dagger = [a(b^\dagger)^*]^\dagger.$$

(d1) ⇒ (e4): Since

$$bb^\dagger a^\dagger a(bb^\dagger a^\dagger a)^\dagger bb^\dagger a^\dagger = (bb^\dagger a^\dagger a(bb^\dagger a^\dagger a)^\dagger bb^\dagger a^\dagger a)a^\dagger = bb^\dagger a^\dagger aa^\dagger = bb^\dagger a^\dagger,$$

$$a(bb^\dagger a^\dagger a)^\dagger bb^\dagger a^\dagger a(bb^\dagger a^\dagger a)^\dagger = a(bb^\dagger a^\dagger a)^\dagger,$$

and $bb^\dagger a^\dagger a(bb^\dagger a^\dagger a)^\dagger$ is self-adjoint, we have that $a(bb^\dagger a^\dagger a)^\dagger \in (bb^\dagger a^\dagger)\{1,2,3\}$. The statement $(bb^\dagger a^\dagger a)^\dagger b \in (b^\dagger a^\dagger a)\{1,2,4\}$ holds because

$$b^\dagger a^\dagger a(bb^\dagger a^\dagger a)^\dagger bb^\dagger a^\dagger a = b^\dagger(bb^\dagger a^\dagger a(bb^\dagger a^\dagger a)^\dagger bb^\dagger a^\dagger a) = b^\dagger bb^\dagger a^\dagger a = b^\dagger a^\dagger a,$$

$$(bb^\dagger a^\dagger a)^\dagger bb^\dagger a^\dagger a(bb^\dagger a^\dagger a)^\dagger b = (bb^\dagger a^\dagger a)^\dagger b,$$

and the element $(bb^\dagger a^\dagger a)^\dagger bb^\dagger a^\dagger a$ is self-adjoint. From $(b^\dagger a^\dagger)^\dagger = a(bb^\dagger a^\dagger a)^\dagger b$, we conclude that the elements $a(bb^\dagger a^\dagger a)^\dagger bb^\dagger a^\dagger$, $b^\dagger a^\dagger a(bb^\dagger a^\dagger a)^\dagger b$ are self–adjoint. Hence, $bb^\dagger a^\dagger, b^\dagger a^\dagger a \in \mathcal{R}^\dagger$, $(bb^\dagger a^\dagger)^\dagger = a(bb^\dagger a^\dagger a)^\dagger$ and $(b^\dagger a^\dagger a)^\dagger = (bb^\dagger a^\dagger a)^\dagger b$. Then, we get $(bb^\dagger a^\dagger)^\dagger b = a(bb^\dagger a^\dagger a)^\dagger b (= (b^\dagger a^\dagger)^\dagger) = a(b^\dagger a^\dagger a)^\dagger$.

(e4) ⇒ (d1): Because

$$bb^\dagger a^\dagger aa^\dagger(bb^\dagger a^\dagger)^\dagger bb^\dagger a^\dagger a = (bb^\dagger a^\dagger(bb^\dagger a^\dagger)^\dagger bb^\dagger a^\dagger)a = bb^\dagger a^\dagger a,$$

$$a^\dagger(bb^\dagger a^\dagger)^\dagger bb^\dagger a^\dagger aa^\dagger(bb^\dagger a^\dagger)^\dagger = a^\dagger(bb^\dagger a^\dagger)^\dagger bb^\dagger a^\dagger(bb^\dagger a^\dagger)^\dagger = a^\dagger(bb^\dagger a^\dagger)^\dagger,$$

and

$$bb^\dagger a^\dagger aa^\dagger(bb^\dagger a^\dagger)^\dagger = bb^\dagger a^\dagger(bb^\dagger a^\dagger)^\dagger \text{ is self-adjoint},$$

we deduce that $a^\dagger(bb^\dagger a^\dagger)^\dagger \in (bb^\dagger a^\dagger a)\{1,2,3\}$. The condition $(bb^\dagger a^\dagger)^\dagger b = a(b^\dagger a^\dagger a)^\dagger$ gives

$$\begin{aligned}(a^\dagger(bb^\dagger a^\dagger)^\dagger bb^\dagger a^\dagger a)^* &= (a^\dagger a(b^\dagger a^\dagger a)^\dagger b^\dagger a^\dagger a)^* = (b^\dagger a^\dagger a)^\dagger b^\dagger a^\dagger aa^\dagger a \\ &= (b^\dagger a^\dagger a)^\dagger b^\dagger a^\dagger a \text{ is self-adjoint}.\end{aligned}$$

Thus, $bb^\dagger a^\dagger a \in \mathcal{R}^\dagger$ and $(bb^\dagger a^\dagger a)^\dagger = a^\dagger(bb^\dagger a^\dagger)^\dagger$. By this equality and (e4), we have

$$a(bb^\dagger a^\dagger a)^\dagger b = aa^\dagger(bb^\dagger a^\dagger)^\dagger b = aa^\dagger a(b^\dagger a^\dagger a)^\dagger = a(b^\dagger a^\dagger a)^\dagger.$$

So, to obtain $(b^\dagger a^\dagger)^\dagger = a(bb^\dagger a^\dagger a)^\dagger b$ it is enough to prove that $(b^\dagger a^\dagger)^\dagger = a(b^\dagger a^\dagger a)^\dagger$. We can easy check that $a(b^\dagger a^\dagger a)^\dagger \in (b^\dagger a^\dagger)\{1,2,3\}$:

$$b^\dagger a^\dagger a(b^\dagger a^\dagger a)^\dagger b^\dagger a^\dagger = (b^\dagger a^\dagger a(b^\dagger a^\dagger a)^\dagger b^\dagger a^\dagger a)a^\dagger = b^\dagger a^\dagger aa^\dagger = b^\dagger a^\dagger,$$

$$a(b^\dagger a^\dagger a)^\dagger b^\dagger a^\dagger a(b^\dagger a^\dagger a)^\dagger = a(b^\dagger a^\dagger a)^\dagger,$$

$$b^\dagger a^\dagger a(b^\dagger a^\dagger a)^\dagger \text{ is self-adjoint}$$

and,by (e4), the element $a(b^\dagger a^\dagger a)^\dagger b^\dagger a^\dagger = (bb^\dagger a^\dagger)^\dagger bb^\dagger a^\dagger$ is self-adjoint. Therefore, $b^\dagger a^\dagger \in \mathcal{R}^\dagger$ and $(b^\dagger a^\dagger)^\dagger = a(b^\dagger a^\dagger a)^\dagger = a(bb^\dagger a^\dagger a)^\dagger b$, i.e. the condition (d1) holds.

(a2) ⇒ (e5): The elementary computations show that $b^*(a^*abb^*)^\dagger \in (a^*ab)\{1,2,3\}$ and $(a^*abb^*)^\dagger a^* \in (abb^*)\{1,2,4\}$ follow from

$$a^*abb^*(a^*abb^*)^\dagger a^*ab = (a^*abb^*(a^*abb^*)^\dagger a^*abb^*)(b^\dagger)^* = a^*abb^*(b^\dagger)^* = a^*ab,$$

$$b^*(a^*abb^*)^\dagger a^*abb^*(a^*abb^*)^\dagger = b^*(a^*abb^*)^\dagger,$$

$$a^*abb^*(a^*abb^*)^\dagger \text{ is self-adjoint}$$

and

$$abb^*(a^*abb^*)^\dagger a^*abb^* = (a^\dagger)^*(a^*abb^*(a^*abb^*)^\dagger a^*abb^*) = (a^\dagger)^*a^*abb^* = abb^*,$$

$$(a^*abb^*)^\dagger a^*abb^*(a^*abb^*)^\dagger a^* = (a^*abb^*)^\dagger a^*,$$

$$(a^*abb^*)^\dagger a^*abb^* \text{ is self-adjoint}.$$

By the hypothesis $(ab)^\dagger = b^*(a^*abb^*)^\dagger a^*$, we observe that the elements $b^*(a^*abb^*)^\dagger a^*ab$, $abb^*(a^*abb^*)^\dagger a^*$ are self-adjoint, i.e. $b^*(a^*abb^*)^\dagger \in (a^*ab)\{4\}$ and $(a^*abb^*)^\dagger a^* \in (abb^*)\{3\}$. Hence, $a^*ab, abb^* \in \mathcal{R}^\dagger$, $(a^*ab)^\dagger = b^*(a^*abb^*)^\dagger$ and $(abb^*)^\dagger = (a^*abb^*)^\dagger a^*$. Then $(a^*ab)^\dagger a^* = b^*(a^*abb^*)^\dagger a^*(= (ab)^\dagger) = b^*(abb^*)^\dagger$.

(e5) ⇒ (a2): In order to prove that $a^*abb^* \in \mathcal{R}^\dagger$, we get first that $(b^\dagger)^*(a^*ab)^\dagger \in (a^*abb^*)\{1,2\}$, by

$$a^*abb^*(b^\dagger)^*(a^*ab)^\dagger a^*abb^* = (a^*ab(a^*ab)^\dagger a^*ab)b^* = a^*abb^*,$$

$$(b^\dagger)^*(a^*ab)^\dagger a^*abb^*(b^\dagger)^*(a^*ab)^\dagger = (b^\dagger)^*(a^*ab)^\dagger a^*ab(a^*ab)^\dagger = (b^\dagger)^*(a^*ab)^\dagger.$$

The equality $a^*abb^*(b^\dagger)^*(a^*ab)^\dagger = a^*ab(a^*ab)^\dagger$ implies that $(b^\dagger)^*(a^*ab)^\dagger \in (a^*abb^*)\{3\}$. From the condition $(a^*ab)^\dagger a^* = b^*(abb^*)^\dagger$, it follows

$$((b^\dagger)^*(a^*ab)^\dagger a^*abb^*)^* = ((b^\dagger)^*b^*(abb^*)^\dagger abb^*)^* = (abb^*)^\dagger abb^*bb^\dagger = (abb^*)^\dagger abb^*$$

implying that $(b^\dagger)^*(a^*ab)^\dagger \in (a^*abb^*)\{4\}$. Therefore, we have $a^*abb^* \in \mathcal{R}^\dagger$ and $(a^*abb^*)^\dagger = (b^\dagger)^*(a^*ab)^\dagger$. This equality and (e5) give

$$b^*(a^*abb^*)^\dagger a^* = b^*(b^\dagger)^*(a^*ab)^\dagger a^* = b^*(b^\dagger)^*b^*(abb^*)^\dagger = b^*(abb^*)^\dagger. \tag{20}$$

To complete the proof we will show that $(ab)^\dagger = b^*(abb^*)^\dagger$. Notice that, by

$$abb^*(abb^*)^\dagger ab = (abb^*(abb^*)^\dagger abb^*)(b^\dagger)^* = abb^*(b^\dagger)^* = ab,$$

$$b^*(abb^*)^\dagger abb^*(abb^*)^\dagger = b^*(abb^*)^\dagger,$$

we get $b^*(abb^*)^\dagger \in (ab)\{1,2\}$. Since $abb^*(abb^*)^\dagger$ is self-adjoint, and, by (e5),

$$b^*(abb^*)^\dagger ab = (a^*ab)^\dagger a^*ab$$

is self-adjoint too, we obtain that $ab \in \mathcal{R}^\dagger$ and $(ab)^\dagger = b^*(abb^*)^\dagger$. Then, from (20), $(ab)^\dagger = b^*(a^*abb^*)^\dagger a^*(= b^*(abb^*)^\dagger = (a^*ab)^\dagger a^*)$.

(b2) ⇒ (e6): To show that $(a^*a)^\dagger b, (a^\dagger)^*bb^* \in \mathcal{R}^\dagger$, let us remark that from

$$(a^*a)^\dagger bb^*[(a^*a)^\dagger bb^*]^\dagger (a^*a)^\dagger b = ((a^*a)^\dagger bb^*[(a^*a)^\dagger bb^*]^\dagger (a^*a)^\dagger bb^*)(b^\dagger)^* = (a^*a)^\dagger bb^*(b^\dagger)^* = (a^*a)^\dagger b,$$

$$b^*[(a^*a)^\dagger bb^*]^\dagger (a^*a)^\dagger bb^*[(a^*a)^\dagger bb^*]^\dagger = b^*[(a^*a)^\dagger bb^*]^\dagger,$$

and

$$(a^\dagger)^*bb^*[(a^*a)^\dagger bb^*]^\dagger a^\dagger (a^\dagger)^*bb^* = a(a^*a)^\dagger bb^*[(a^*a)^\dagger bb^*]^\dagger (a^*a)^\dagger bb^* = a(a^*a)^\dagger bb^* = (a^\dagger)^*bb^*,$$

$$[(a^*a)^\dagger bb^*]^\dagger a^\dagger (a^\dagger)^*bb^*[(a^*a)^\dagger bb^*]^\dagger a^\dagger = [(a^*a)^\dagger bb^*]^\dagger (a^*a)^\dagger bb^*[(a^*a)^\dagger bb^*]^\dagger a^\dagger = [(a^*a)^\dagger bb^*]^\dagger a^\dagger,$$

we get $b^*[(a^*a)^\dagger bb^*]^\dagger \in [(a^*a)^\dagger b]\{1,2\}$ and $[(a^*a)^\dagger bb^*]^\dagger a^\dagger \in [(a^\dagger)^*bb^*]\{1,2\}$. Obviously, the elements $(a^*a)^\dagger bb^*[(a^*a)^\dagger bb^*]^\dagger$ and $[(a^*a)^\dagger bb^*]^\dagger a^\dagger (a^\dagger)^*bb^* = [(a^*a)^\dagger bb^*]^\dagger (a^*a)^\dagger bb^*$ are self-adjoint. From the hypothesis $[(a^\dagger)^*b]^\dagger = b^*[(a^*a)^\dagger bb^*]^\dagger a^\dagger$ we have that $b^*[(a^*a)^\dagger bb^*]^\dagger (a^*a)^\dagger b = b^*[(a^*a)^\dagger bb^*]^\dagger a^\dagger (a^\dagger)^*b$ and $(a^\dagger)^*bb^*[(a^*a)^\dagger bb^*]^\dagger a^\dagger$

are self-adjoint elements. Thus, $(a^*a)^\dagger b, (a^\dagger)^* bb^* \in \mathcal{R}^\dagger$, $[(a^*a)^\dagger b]^\dagger = b^*[(a^*a)^\dagger bb^*]^\dagger$ and $[(a^\dagger)^* bb^*]^\dagger = [(a^*a)^\dagger bb^*]^\dagger a^\dagger$. Now we deduce that $[(a^*a)^\dagger b]^\dagger a^\dagger = b^*[(a^*a)^\dagger bb^*]^\dagger a^\dagger (= [(a^\dagger)^* b]^\dagger) = b^*[(a^\dagger)^* bb^*]^\dagger$.

(e6) ⇒ (b2): To prove the condition $(a^*a)^\dagger bb^* \in \mathcal{R}^\dagger$ we observe that $(b^\dagger)^*[(a^*a)^\dagger b]^\dagger \in [(a^*a)^\dagger bb^*]\{1,2,3\}$ by

$$(a^*a)^\dagger bb^*(b^\dagger)^*[(a^*a)^\dagger b]^\dagger (a^*a)^\dagger bb^* = ((a^*a)^\dagger b[(a^*a)^\dagger b]^\dagger (a^*a)^\dagger b)b^* = (a^*a)^\dagger bb^*,$$

$$(b^\dagger)^*[(a^*a)^\dagger b]^\dagger (a^*a)^\dagger bb^*(b^\dagger)^*[(a^*a)^\dagger b]^\dagger = (b^\dagger)^*[(a^*a)^\dagger b]^\dagger (a^*a)^\dagger b[(a^*a)^\dagger b]^\dagger = (b^\dagger)^*[(a^*a)^\dagger b]^\dagger,$$

$$(a^*a)^\dagger bb^*(b^\dagger)^*[(a^*a)^\dagger b]^\dagger = (a^*a)^\dagger b[(a^*a)^\dagger b]^\dagger \text{ is self}-\text{adjoint.}$$

Using the equality $[(a^*a)^\dagger b]^\dagger a^\dagger = b^*[(a^\dagger)^* bb^*]^\dagger$, we obtain

$$\begin{aligned}((b^\dagger)^*[(a^*a)^\dagger b]^\dagger (a^*a)^\dagger bb^*)^* &= ((b^\dagger)^*[(a^*a)^\dagger b]^\dagger a^\dagger (a^\dagger)^* bb^*)^* = ((b^\dagger)^* b^*[(a^\dagger)^* bb^*]^\dagger (a^\dagger)^* bb^*)^* \\ &= [(a^\dagger)^* bb^*]^\dagger (a^\dagger)^* bb^* bb^\dagger = [(a^\dagger)^* bb^*]^\dagger (a^\dagger)^* bb^*,\end{aligned}$$

that is $(b^\dagger)^*[(a^*a)^\dagger b]^\dagger \in [(a^*a)^\dagger bb^*]\{4\}$. So, we get $(a^*a)^\dagger bb^* \in \mathcal{R}^\dagger$ and $[(a^*a)^\dagger bb^*]^\dagger = (b^\dagger)^*[(a^*a)^\dagger b]^\dagger$. By this equality and (e6),

$$b^*[(a^*a)^\dagger bb^*]^\dagger a^\dagger = b^*(b^\dagger)^*[(a^*a)^\dagger b]^\dagger a^\dagger = b^*(b^\dagger)^* b^*[(a^\dagger)^* bb^*]^\dagger = b^*[(a^\dagger)^* bb^*]^\dagger.$$

If we show that $(a^\dagger)^* b \in \mathcal{R}^\dagger$ and $[(a^\dagger)^* b]^\dagger = b^*[(a^\dagger)^* bb^*]^\dagger$, it follows that $[(a^\dagger)^* b]^\dagger = b^*[(a^*a)^\dagger bb^*]^\dagger a^\dagger$. We can see that $b^*[(a^\dagger)^* bb^*]^\dagger \in [(a^\dagger)^* b]\{1,2,3\}$, by

$$(a^\dagger)^* bb^*[(a^\dagger)^* bb^*]^\dagger (a^\dagger)^* b = ((a^\dagger)^* bb^*[(a^\dagger)^* bb^*]^\dagger (a^\dagger)^* bb^*)(b^\dagger)^* = (a^\dagger)^* bb^*(b^\dagger)^* = (a^\dagger)^* b,$$

$$b^*[(a^\dagger)^* bb^*]^\dagger (a^\dagger)^* bb^*[(a^\dagger)^* bb^*]^\dagger = b^*[(a^\dagger)^* bb^*]^\dagger,$$

$$(a^\dagger)^* bb^*[(a^\dagger)^* bb^*]^\dagger \text{ is self}-\text{adjoint.}$$

The condition $b^*[(a^\dagger)^* bb^*]^\dagger \in [(a^\dagger)^* b]\{4\}$ holds, because (e6) gives

$$b^*[(a^\dagger)^* bb^*]^\dagger (a^\dagger)^* b = [(a^*a)^\dagger b]^\dagger a^\dagger (a^\dagger)^* b = [(a^*a)^\dagger b]^\dagger (a^*a)^\dagger b \text{ is self}-\text{adjoint.}$$

Hence, $(a^\dagger)^* b \in \mathcal{R}^\dagger$ and $[(a^\dagger)^* b]^\dagger = b^*[(a^\dagger)^* bb^*]^\dagger = b^*[(a^*a)^\dagger bb^*]^\dagger a^\dagger$.

(c2) ⇒ (e7): Notice that we have $b^\dagger[a^*a(bb^*)^\dagger]^\dagger \in [a^*a(b^\dagger)^*]\{1,2,3\}$ and $[a^*a(bb^*)^\dagger]^\dagger a^* \in [a(bb^*)^\dagger]\{1,2,4\}$, from

$$a^*a(b^\dagger)^* b^\dagger[a^*a(bb^*)^\dagger]^\dagger a^*a(b^\dagger)^* = (a^*a(bb^*)^\dagger[a^*a(bb^*)^\dagger]^\dagger a^*a(bb^*)^\dagger)b = a^*a(bb^*)^\dagger b = a^*a(b^\dagger)^*,$$

$$b^\dagger[a^*a(bb^*)^\dagger]^\dagger a^*a(b^\dagger)^* b^\dagger[a^*a(bb^*)^\dagger]^\dagger = b^\dagger[a^*a(bb^*)^\dagger]^\dagger a^*a(bb^*)^\dagger[a^*a(bb^*)^\dagger]^\dagger = b^\dagger[a^*a(bb^*)^\dagger]^\dagger,$$

$$a^*a(b^\dagger)^* b^\dagger[a^*a(bb^*)^\dagger]^\dagger = a^*a(bb^*)^\dagger[a^*a(bb^*)^\dagger]^\dagger \text{ is self}-\text{adjoint}$$

and

$$a(bb^*)^\dagger[a^*a(bb^*)^\dagger]^\dagger a^*a(bb^*)^\dagger = (a^\dagger)^*(a^*a(bb^*)^\dagger[a^*a(bb^*)^\dagger]^\dagger a^*a(bb^*)^\dagger) = (a^\dagger)^* a^*a(bb^*)^\dagger = a(bb^*)^\dagger,$$

$$[a^*a(bb^*)^\dagger]^\dagger a^*a(bb^*)^\dagger[a^*a(bb^*)^\dagger]^\dagger a^* = [a^*a(bb^*)^\dagger]^\dagger a^*,$$

$$[a^*a(bb^*)^\dagger]^\dagger a^*a(bb^*)^\dagger \text{ is self}-\text{adjoint.}$$

The assumption $[a(b^\dagger)^*]^\dagger = b^\dagger[a^*a(bb^*)^\dagger]^\dagger a^*$ implies that

$$b^\dagger[a^*a(bb^*)^\dagger]^\dagger a^*a(b^\dagger)^* \text{ is self}-\text{adjoint}$$

and

$$a(bb^*)^\dagger[a^*a(bb^*)^\dagger]^\dagger a^* = a(b^\dagger)^*b^\dagger[a^*a(bb^*)^\dagger]^\dagger a^* \text{ is self}-\text{adjoint},$$

i.e. $b^\dagger[a^*a(bb^*)^\dagger]^\dagger \in [a^*a(b^\dagger)^*]\{4\}$ and $[a^*a(bb^*)^\dagger]^\dagger a^* \in [a(bb^*)^\dagger]\{3\}$. Therefore, we conclude $a^*a(b^\dagger)^*, a(bb^*)^\dagger \in \mathcal{R}^\dagger$, $[a^*a(b^\dagger)^*]^\dagger = b^\dagger[a^*a(bb^*)^\dagger]^\dagger$ and $[a(bb^*)^\dagger]^\dagger = [a^*a(bb^*)^\dagger]^\dagger a^*$. Now, we have $[a^*a(b^\dagger)^*]^\dagger a^* = b^\dagger[a^*a(bb^*)^\dagger]^\dagger a^*(= [a(b^\dagger)^*]^\dagger) = b^\dagger[a(bb^*)^\dagger]^\dagger$.

(e7) $\Rightarrow$ (c2): It is easy to check that $[a(bb^*)^\dagger]^\dagger(a^\dagger)^* \in [a^*a(bb^*)^\dagger]\{1,2,4\}$:

$$a^*a(bb^*)^\dagger[a(bb^*)^\dagger]^\dagger(a^\dagger)^*a^*a(bb^*)^\dagger = a^*(a(bb^*)^\dagger[a(bb^*)^\dagger]^\dagger a(bb^*)^\dagger) = a^*a(bb^*)^\dagger,$$

$$[a(bb^*)^\dagger]^\dagger(a^\dagger)^*a^*a(bb^*)^\dagger[a(bb^*)^\dagger]^\dagger(a^\dagger)^* = [a(bb^*)^\dagger]^\dagger a(bb^*)^\dagger[a(bb^*)^\dagger]^\dagger(a^\dagger)^* = [a(bb^*)^\dagger]^\dagger(a^\dagger)^*,$$

$$[a(bb^*)^\dagger]^\dagger(a^\dagger)^*a^*a(bb^*)^\dagger = [a(bb^*)^\dagger]^\dagger a(bb^*)^\dagger \text{ is self}-\text{adjoint}.$$

Using $[a^*a(b^\dagger)^*]^\dagger a^* = b^\dagger[a(bb^*)^\dagger]^\dagger$, we obtain

$$\begin{aligned}(a^*a(bb^*)^\dagger[a(bb^*)^\dagger]^\dagger(a^\dagger)^*)^* &= (a^*a(b^\dagger)^*b^\dagger[a(bb^*)^\dagger]^\dagger(a^\dagger)^*)^* = (a^*a(b^\dagger)^*[a^*a(b^\dagger)^*]^\dagger a^*(a^\dagger)^*)^* \\ &= a^\dagger aa^*a(b^\dagger)^*[a^*a(b^\dagger)^*]^\dagger = a^*a(b^\dagger)^*[a^*a(b^\dagger)^*]^\dagger \text{ is self}-\text{adjoint}.\end{aligned}$$

Hence, we have $a^*a(bb^*)^\dagger \in \mathcal{R}^\dagger$ and $[a^*a(bb^*)^\dagger]^\dagger = [a(bb^*)^\dagger]^\dagger(a^\dagger)^*$. Since, by this equality and (e7),

$$b^\dagger[a^*a(bb^*)^\dagger]^\dagger a^* = b^\dagger[a(bb^*)^\dagger]^\dagger(a^\dagger)^*a^* = [a^*a(b^\dagger)^*]^\dagger a^*(a^\dagger)^*a^* = [a^*a(b^\dagger)^*]^\dagger a^*,$$

in order to show that $a(b^\dagger)^* \in \mathcal{R}^\dagger$ and $[a(b^\dagger)^*]^\dagger = b^\dagger[a^*a(bb^*)^\dagger]^\dagger a^*$, we will prove that $[a(b^\dagger)^*]^\dagger = [a^*a(b^\dagger)^*]^\dagger a^*$. Indeed, $[a^*a(b^\dagger)^*]^\dagger a^* \in [a(b^\dagger)^*]\{1,2,4\}$ follows from

$$a(b^\dagger)^*[a^*a(b^\dagger)^*]^\dagger a^*a(b^\dagger)^* = (a^\dagger)^*(a^*a(b^\dagger)^*[a^*a(b^\dagger)^*]^\dagger a^*a(b^\dagger)^*) = (a^\dagger)^*a^*a(b^\dagger)^* = a(b^\dagger)^*,$$

$$[a^*a(b^\dagger)^*]^\dagger a^*a(b^\dagger)^*[a^*a(b^\dagger)^*]^\dagger a^* = [a^*a(b^\dagger)^*]^\dagger a^*,$$

$$[a^*a(b^\dagger)^*]^\dagger a^*a(b^\dagger)^* \text{ is self}-\text{adjoint}.$$

By (e7),

$$a(b^\dagger)^*[a^*a(b^\dagger)^*]^\dagger a^* = a(b^\dagger)^*b^\dagger[a(bb^*)^\dagger]^\dagger = a(bb^*)^\dagger[a(bb^*)^\dagger]^\dagger \text{ is self}-\text{adjoint}.$$

So, $a(b^\dagger)^* \in \mathcal{R}^\dagger$ and $[a(b^\dagger)^*]^\dagger = [a^*a(b^\dagger)^*]^\dagger a^* = b^\dagger[a^*a(bb^*)^\dagger]^\dagger a^*$, that is (c2) is satisfied.

(d2) $\Rightarrow$ (e8): First, let us show that $(bb^*)^\dagger a^\dagger, b^\dagger(a^*a)^\dagger \in \mathcal{R}^\dagger$. From

$$\begin{aligned}(bb^*)^\dagger a^\dagger(a^\dagger)^*[(bb^*)^\dagger(a^*a)^\dagger]^\dagger(bb^*)^\dagger a^\dagger &= ((bb^*)^\dagger(a^*a)^\dagger[(bb^*)^\dagger(a^*a)^\dagger]^\dagger(bb^*)^\dagger(a^*a)^\dagger)a^* \\ &= (bb^*)^\dagger(a^*a)^\dagger a^* = (bb^*)^\dagger a^\dagger,\end{aligned}$$

$$\begin{aligned}(a^\dagger)^*[(bb^*)^\dagger(a^*a)^\dagger]^\dagger(bb^*)^\dagger a^\dagger(a^\dagger)^*[(bb^*)^\dagger(a^*a)^\dagger]^\dagger &= (a^\dagger)^*[(bb^*)^\dagger(a^*a)^\dagger]^\dagger(bb^*)^\dagger(a^*a)^\dagger[(bb^*)^\dagger(a^*a)^\dagger]^\dagger \\ &= (a^\dagger)^*[(bb^*)^\dagger(a^*a)^\dagger]^\dagger,\end{aligned}$$

$$(bb^*)^\dagger a^\dagger (a^\dagger)^*[(bb^*)^\dagger (a^*a)^\dagger]^\dagger = (bb^*)^\dagger (a^*a)^\dagger [(bb^*)^\dagger (a^*a)^\dagger]^\dagger \text{ is self}-\text{adjoint},$$

we deduce that $(a^\dagger)^*[(bb^*)^\dagger(a^*a)^\dagger]^\dagger \in [(bb^*)^\dagger a^\dagger]\{1,2,3\}$. The statement $[(bb^*)^\dagger(a^*a)^\dagger]^\dagger(b^\dagger)^* \in [b^\dagger(a^*a)^\dagger]\{1,2,4\}$ is a simple consequence of the equalities

$$\begin{aligned} b^\dagger(a^*a)^\dagger[(bb^*)^\dagger(a^*a)^\dagger]^\dagger(b^\dagger)^*b^\dagger(a^*a)^\dagger &= b^*((bb^*)^\dagger(a^*a)^\dagger[(bb^*)^\dagger(a^*a)^\dagger]^\dagger(bb^*)^\dagger(a^*a)^\dagger) \\ &= b^*(bb^*)^\dagger(a^*a)^\dagger = b^\dagger(a^*a)^\dagger, \end{aligned}$$

$$\begin{aligned} [(bb^*)^\dagger(a^*a)^\dagger]^\dagger(b^\dagger)^*b^\dagger(a^*a)^\dagger[(bb^*)^\dagger(a^*a)^\dagger]^\dagger(b^\dagger)^* &= [(bb^*)^\dagger(a^*a)^\dagger]^\dagger(bb^*)^\dagger(a^*a)^\dagger[(bb^*)^\dagger(a^*a)^\dagger]^\dagger(b^\dagger)^* \\ &= [(bb^*)^\dagger(a^*a)^\dagger]^\dagger(b^\dagger)^*, \end{aligned}$$

$$[(bb^*)^\dagger(a^*a)^\dagger]^\dagger(b^\dagger)^*b^\dagger(a^*a)^\dagger = [(bb^*)^\dagger(a^*a)^\dagger]^\dagger(bb^*)^\dagger(a^*a)^\dagger \text{ is self}-\text{adjoint}.$$

The hypothesis $(b^\dagger a^\dagger)^\dagger = (a^\dagger)^*[(bb^*)^\dagger(a^*a)^\dagger]^\dagger(b^\dagger)^*$ gives that the elements

$$(a^\dagger)^*[(bb^*)^\dagger(a^*a)^\dagger]^\dagger(bb^*)^\dagger a^\dagger = (a^\dagger)^*[(bb^*)^\dagger(a^*a)^\dagger]^\dagger(b^\dagger)^*b^\dagger a^\dagger$$

and

$$b^\dagger(a^*a)^\dagger[(bb^*)^\dagger(a^*a)^\dagger]^\dagger(b^\dagger)^* = b^\dagger a^\dagger(a^\dagger)^*[(bb^*)^\dagger(a^*a)^\dagger]^\dagger(b^\dagger)^*$$

are self-adjoint, i.e. we obtain that $(a^\dagger)^*[(bb^*)^\dagger(a^*a)^\dagger]^\dagger \in [(bb^*)^\dagger a^\dagger]\{4\}$ and $[(bb^*)^\dagger(a^*a)^\dagger]^\dagger(b^\dagger)^* \in [b^\dagger(a^*a)^\dagger]\{3\}$. Consequently, $(bb^*)^\dagger a^\dagger, b^\dagger(a^*a)^\dagger \in \mathcal{R}^\dagger$, $[(bb^*)^\dagger a^\dagger]^\dagger = (a^\dagger)^*[(bb^*)^\dagger(a^*a)^\dagger]^\dagger$ and $[b^\dagger(a^*a)^\dagger]^\dagger = [(bb^*)^\dagger(a^*a)^\dagger]^\dagger(b^\dagger)^*$. Then

$$[(bb^*)^\dagger a^\dagger]^\dagger(b^\dagger)^* = (a^\dagger)^*[(bb^*)^\dagger(a^*a)^\dagger]^\dagger(b^\dagger)^*(= (b^\dagger a^\dagger)^\dagger) = (a^\dagger)^*[b^\dagger(a^*a)^\dagger]^\dagger \tag{21}$$

and, by Theorem 1.1, $(a^\dagger)^*(bb^*)^\dagger = [(bb^*)^\dagger a^\dagger]^*, (a^*a)^\dagger(b^\dagger)^* = [b^\dagger(a^*a)^\dagger]^* \in \mathcal{R}^\dagger$. Applying involution to (21), we have $b^\dagger[(a^\dagger)^*(bb^*)^\dagger]^\dagger = [(a^*a)^\dagger(b^\dagger)^*]^\dagger a^\dagger$ and the condition (e8) holds.

(e8) ⇒ (d2): By the elementary computations, we get

$$\begin{aligned} (bb^*)^\dagger(a^*a)^\dagger a^*[(bb^*)^\dagger a^\dagger]^\dagger(bb^*)^\dagger(a^*a)^\dagger &= ((bb^*)^\dagger a^\dagger[(bb^*)^\dagger a^\dagger]^\dagger(bb^*)^\dagger a^\dagger)(a^\dagger)^* \\ &= (bb^*)^\dagger a^\dagger(a^\dagger)^* = (bb^*)^\dagger(a^*a)^\dagger, \end{aligned}$$

$$\begin{aligned} a^*[(bb^*)^\dagger a^\dagger]^\dagger(bb^*)^\dagger(a^*a)^\dagger a^*[(bb^*)^\dagger a^\dagger]^\dagger &= a^*[(bb^*)^\dagger a^\dagger]^\dagger(bb^*)^\dagger a^\dagger[(bb^*)^\dagger a^\dagger]^\dagger \\ &= a^*[(bb^*)^\dagger a^\dagger]^\dagger, \end{aligned}$$

$$(bb^*)^\dagger(a^*a)^\dagger a^*[(bb^*)^\dagger a^\dagger]^\dagger = (bb^*)^\dagger a^\dagger[(bb^*)^\dagger a^\dagger]^\dagger \text{ is self}-\text{adjoint},$$

which yield $a^*[(bb^*)^\dagger a^\dagger]^\dagger \in [(bb^*)^\dagger(a^*a)^\dagger]\{1,2,3\}$. Applying involution to the condition $b^\dagger[(a^\dagger)^*(bb^*)^\dagger]^\dagger = [(a^*a)^\dagger(b^\dagger)^*]^\dagger a^\dagger$, we obtain

$$[(bb^*)^\dagger a^\dagger]^\dagger(b^\dagger)^* = (a^\dagger)^*[b^\dagger(a^*a)^\dagger]^\dagger \tag{22}$$

and

$$\begin{aligned}(a^*[(bb^*)^\dagger a^\dagger]^\dagger (bb^*)^\dagger (a^*a)^\dagger)^* &= (a^*[(bb^*)^\dagger a^\dagger]^\dagger (b^\dagger)^* b^\dagger (a^*a)^\dagger)^* \\ &= (a^*(a^\dagger)^*[b^\dagger (a^*a)^\dagger]^\dagger b^\dagger (a^*a)^\dagger)^* \\ &= [b^\dagger (a^*a)^\dagger]^\dagger b^\dagger (a^*a)^\dagger a^\dagger a \\ &= [b^\dagger (a^*a)^\dagger]^\dagger b^\dagger a^\dagger (a^\dagger)^* a^\dagger a \\ &= [b^\dagger (a^*a)^\dagger]^\dagger b^\dagger a^\dagger (a^\dagger)^* \\ &= [b^\dagger (a^*a)^\dagger]^\dagger b^\dagger (a^*a)^\dagger \text{ is self}-\text{adjoint.}\end{aligned}$$

Thus, $(bb^*)^\dagger (a^*a)^\dagger \in \mathcal{R}^\dagger$ and $[(bb^*)^\dagger (a^*a)^\dagger]^\dagger = a^*[(bb^*)^\dagger a^\dagger]^\dagger$. This equality and (22) give that

$$\begin{aligned}(a^\dagger)^*[(bb^*)^\dagger (a^*a)^\dagger]^\dagger (b^\dagger)^* &= (a^\dagger)^* a^*[(bb^*)^\dagger a^\dagger]^\dagger (b^\dagger)^* = aa^\dagger (a^\dagger)^*[b^\dagger (a^*a)^\dagger]^\dagger \\ &= (a^\dagger)^*[b^\dagger (a^*a)^\dagger]^\dagger.\end{aligned}$$

Now, to prove $(b^\dagger a^\dagger)^\dagger = (a^\dagger)^*[(bb^*)^\dagger (a^*a)^\dagger]^\dagger (b^\dagger)^*$ it is enough to check that $(b^\dagger a^\dagger)^\dagger = (a^\dagger)^*[b^\dagger (a^*a)^\dagger]^\dagger$. We show that $(a^\dagger)^*[b^\dagger (a^*a)^\dagger]^\dagger \in (b^\dagger a^\dagger)\{1,2,3\}$ by

$$\begin{aligned}b^\dagger a^\dagger (a^\dagger)^*[b^\dagger (a^*a)^\dagger]^\dagger b^\dagger a^\dagger &= (b^\dagger (a^*a)^\dagger [b^\dagger (a^*a)^\dagger]^\dagger b^\dagger (a^*a)^\dagger) a^* \\ &= b^\dagger (a^*a)^\dagger a^* = b^\dagger a^\dagger,\end{aligned}$$

$$\begin{aligned}(a^\dagger)^*[b^\dagger (a^*a)^\dagger]^\dagger b^\dagger a^\dagger (a^\dagger)^*[b^\dagger (a^*a)^\dagger]^\dagger &= (a^\dagger)^*[b^\dagger (a^*a)^\dagger]^\dagger b^\dagger (a^*a)^\dagger [b^\dagger (a^*a)^\dagger]^\dagger \\ &= (a^\dagger)^*[b^\dagger (a^*a)^\dagger]^\dagger,\end{aligned}$$

$$b^\dagger a^\dagger (a^\dagger)^*[b^\dagger (a^*a)^\dagger]^\dagger = b^\dagger (a^*a)^\dagger [b^\dagger (a^*a)^\dagger]^\dagger \text{ is self}-\text{adjoint.}$$

From (22),

$$(a^\dagger)^*[b^\dagger (a^*a)^\dagger]^\dagger b^\dagger a^\dagger = [(bb^*)^\dagger a^\dagger]^\dagger (b^\dagger)^* b^\dagger a^\dagger = [(bb^*)^\dagger a^\dagger]^\dagger (bb^*)^\dagger a^\dagger,$$

that is $(a^\dagger)^*[b^\dagger (a^*a)^\dagger]^\dagger \in (b^\dagger a^\dagger)\{4\}$. So, we obtain that $b^\dagger a^\dagger \in \mathcal{R}^\dagger$ and $(b^\dagger a^\dagger)^\dagger = (a^\dagger)^*[b^\dagger (a^*a)^\dagger]^\dagger = (a^\dagger)^*[(bb^*)^\dagger (a^*a)^\dagger]^\dagger (b^\dagger)^*$.

(a2) ⇒ (e9): From

$$aa^*abb^*bb^\dagger (a^*abb^*)^\dagger a^\dagger aa^*abb^*b = a(a^*abb^*(a^*abb^*)^\dagger a^*abb^*)b = aa^*abb^*b,$$

$$b^\dagger (a^*abb^*)^\dagger a^\dagger aa^*abb^*bb^\dagger (a^*abb^*)^\dagger a^\dagger = b^\dagger (a^*abb^*)^\dagger a^*abb^*(a^*abb^*)^\dagger a^\dagger = b^\dagger (a^*abb^*)^\dagger a^\dagger,$$

we conclude that $b^\dagger (a^*abb^*)^\dagger a^\dagger \in (aa^*abb^*b)\{1,2\}$. By the equality

$$(aa^*abb^*bb^\dagger (a^*abb^*)^\dagger a^\dagger)^* = (aa^*abb^*(a^*abb^*)^\dagger a^\dagger)^* = (a^\dagger)^* a^*abb^*(a^*abb^*)^\dagger a^* = abb^*(a^*abb^*)^\dagger a^*,$$

$$(b^\dagger (a^*abb^*)^\dagger a^\dagger aa^*abb^*b)^* = (b^\dagger (a^*abb^*)^\dagger a^*abb^*b)^* = b^*(a^*abb^*)^\dagger a^*abb^*(b^\dagger)^* = b^*(a^*abb^*)^\dagger a^*ab$$

and the assumption $(ab)^\dagger = b^*(a^*abb^*)^\dagger a^*$, we observe that $b^\dagger (a^*abb^*)^\dagger a^\dagger \in (aa^*abb^*b)\{3,4\}$. Hence, $aa^*abb^*b \in \mathcal{R}^\dagger$ and $(aa^*abb^*b)^\dagger = b^\dagger (a^*abb^*)^\dagger a^\dagger$.

(e9) ⇒ (a2): We can get that $b^*(a^*abb^*)^\dagger a^* \in (ab)\{1,2\}$ in the following way

$$abb^*(a^*abb^*)^\dagger a^*ab = (a^\dagger)^*(a^*abb^*(a^*abb^*)^\dagger a^*abb^*)(b^\dagger)^* = (a^\dagger)^*a^*abb^*(b^\dagger)^* = ab,$$

$$b^*(a^*abb^*)^\dagger a^*abb^*(a^*abb^*)^\dagger a^* = b^*(a^*abb^*)^\dagger a^*.$$

From the hypothesis $(aa^*abb^*b)^\dagger = b^\dagger(a^*abb^*)^\dagger a^\dagger$ we obtain

$$\begin{aligned}(abb^*(a^*abb^*)^\dagger a^*)^* &= ((a^\dagger)^*a^*abb^*(a^*abb^*)^\dagger a^*)^* = aa^*abb^*(a^*abb^*)^\dagger a^\dagger \\ &= aa^*abb^*bb^\dagger(a^*abb^*)^\dagger a^\dagger \text{ is self}-\text{adjoint}\end{aligned}$$

and

$$\begin{aligned}(b^*(a^*abb^*)^\dagger a^*ab)^* &= (b^*(a^*abb^*)^\dagger a^*abb^*(b^\dagger)^*)^* = b^\dagger(a^*abb^*)^\dagger a^*abb^*b \\ &= b^\dagger(a^*abb^*)^\dagger a^\dagger aa^*abb^*b \text{ is self}-\text{adjoint.}\end{aligned}$$

Thus, $ab \in \mathcal{R}^\dagger$ and $(ab)^\dagger = b^*(a^*abb^*)^\dagger a^*$, i.e. the statements (a2) is satisfied.

(f1) ⇒ (f2): First, we will prove that $a^\dagger abb^* \in \mathcal{R}^\dagger$. From

$$a^\dagger abb^*(b^\dagger)^*b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger abb^* = (a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger)bb^* = a^\dagger abb^\dagger bb^* = a^\dagger abb^*,$$

$$(b^\dagger)^*b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger abb^*(b^\dagger)^*b^\dagger(a^\dagger abb^\dagger)^\dagger = (b^\dagger)^*b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger = (b^\dagger)^*b^\dagger(a^\dagger abb^\dagger)^\dagger,$$

$$a^\dagger abb^*(b^\dagger)^*b^\dagger(a^\dagger abb^\dagger)^\dagger = a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger,$$

we have that $(b^\dagger)^*b^\dagger(a^\dagger abb^\dagger)^\dagger \in (a^\dagger abb^*)\{1,2,3\}$. Using the assumption $(a^\dagger ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger$, we get $b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger ab$ is self-adjoint and

$$\begin{aligned}(b^\dagger)^*b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger abb^* &= (bb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger)^* = (a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger bb^\dagger \\ &= (a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger \text{ is self}-\text{adjoint.}\end{aligned}$$

Therefore, $a^\dagger abb^* \in \mathcal{R}^\dagger$ and $(a^\dagger abb^*)^\dagger = (b^\dagger)^*b^\dagger(a^\dagger abb^\dagger)^\dagger$. By this equality and (f1) we obtain

$$(a^\dagger ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger = b^*(b^\dagger)^*b^\dagger(a^\dagger abb^\dagger)^\dagger = b^*(a^\dagger abb^*)^\dagger.$$

In the same way from the equalities

$$a^*abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger(a^\dagger)^*a^*abb^\dagger = a^*a(a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger) = a^*aa^\dagger abb^\dagger = a^*abb^\dagger,$$

$$(a^\dagger abb^\dagger)^\dagger a^\dagger(a^\dagger)^*a^*abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger(a^\dagger)^* = (a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger(a^\dagger)^* = (a^\dagger abb^\dagger)^\dagger a^\dagger(a^\dagger)^*,$$

$$(a^\dagger abb^\dagger)^\dagger a^\dagger(a^\dagger)^*a^*abb^\dagger = (a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger,$$

we deduce $(a^\dagger abb^\dagger)^\dagger a^\dagger(a^\dagger)^* \in (a^*abb^\dagger)\{1,2,4\}$. The hypothesis $(abb^\dagger)^\dagger = (a^\dagger abb^\dagger)^\dagger a^\dagger$ implies that $abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ is self-adjoint and then

$$\begin{aligned}a^*abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger(a^\dagger)^* &= (a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger a)^* = a^\dagger aa^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger \\ &= a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger \text{ is self}-\text{adjoint.}\end{aligned}$$

Thus, we get that $a^*abb^\dagger \in \mathcal{R}^\dagger$, $(a^*abb^\dagger)^\dagger = (a^\dagger abb^\dagger)^\dagger a^\dagger (a^\dagger)^*$ and, by (f1),

$$(abb^\dagger)^\dagger = (a^\dagger abb^\dagger)^\dagger a^\dagger = (a^\dagger abb^\dagger)^\dagger a^\dagger (a^\dagger)^* a^* = (a^*abb^\dagger)^\dagger a^*.$$

So, the condition (f2) is satisfied.

(f2) ⇒ (f1): Since

$$a^\dagger abb^\dagger bb^*(a^\dagger abb^*)^\dagger a^\dagger abb^\dagger = (a^\dagger abb^*(a^\dagger abb^*)^\dagger a^\dagger abb^*)(b^\dagger)^* b^\dagger = a^\dagger abb^*(b^\dagger)^* b^\dagger = a^\dagger abb^\dagger,$$

$$bb^*(a^\dagger abb^*)^\dagger a^\dagger abb^\dagger bb^*(a^\dagger abb^*)^\dagger = bb^*(a^\dagger abb^*)^\dagger a^\dagger abb^*(a^\dagger abb^*)^\dagger = bb^*(a^\dagger abb^*)^\dagger,$$

$$a^\dagger abb^\dagger bb^*(a^\dagger abb^*)^\dagger = a^\dagger abb^*(a^\dagger abb^*)^\dagger \text{ is self}-\text{adjoint},$$

we conclude that $bb^*(a^\dagger abb^*)^\dagger \in (a^\dagger abb^\dagger)\{1,2,3\}$. By the equality $(a^\dagger ab)^\dagger = b^*(a^\dagger abb^*)^\dagger$, we have that $b^*(a^\dagger abb^*)^\dagger a^\dagger ab$ is self-adjoint and then

$$\begin{aligned} bb^*(a^\dagger abb^*)^\dagger a^\dagger abb^\dagger &= ((b^\dagger)^* b^*(a^\dagger abb^*)^\dagger a^\dagger abb^*)^* = (a^\dagger abb^*)^\dagger a^\dagger abb^* bb^\dagger \\ &= (a^\dagger abb^*)^\dagger a^\dagger abb^* \text{ is self}-\text{adjoint}. \end{aligned}$$

Hence, $a^\dagger abb^\dagger \in \mathcal{R}^\dagger$ and $(a^\dagger abb^\dagger)^\dagger = bb^*(a^\dagger abb^*)^\dagger$. Now, by (f2) and the last equality,

$$(a^\dagger ab)^\dagger = b^*(a^\dagger abb^*)^\dagger = b^\dagger bb^*(a^\dagger abb^*)^\dagger = b^\dagger (a^\dagger abb^\dagger)^\dagger.$$

Similarly, from the equalities

$$a^\dagger abb^\dagger (a^*abb^\dagger)^\dagger a^*aa^\dagger abb^\dagger = a^\dagger (a^\dagger)^*(a^*abb^\dagger (a^*abb^\dagger)^\dagger a^*abb^\dagger) = a^\dagger (a^\dagger)^* a^*abb^\dagger = a^\dagger abb^\dagger,$$

$$(a^*abb^\dagger)^\dagger a^*aa^\dagger abb^\dagger (a^*abb^\dagger)^\dagger a^*a = (a^*abb^\dagger)^\dagger a^*abb^\dagger (a^*abb^\dagger)^\dagger a^*a = (a^*abb^\dagger)^\dagger a^*a$$

$$(a^*abb^\dagger)^\dagger a^*aa^\dagger abb^\dagger = (a^*abb^\dagger)^\dagger a^*abb^\dagger \text{ is self}-\text{adjoint},$$

we obtain that $(a^*abb^\dagger)^\dagger a^*a \in (a^\dagger abb^\dagger)\{1,2,4\}$. Using the condition $(abb^\dagger)^\dagger = (a^*abb^\dagger)^\dagger a^*$, the element $abb^\dagger (a^*abb^\dagger)^\dagger a^*$ is self-adjoint and now

$$\begin{aligned} a^\dagger abb^\dagger (a^*abb^\dagger)^\dagger a^*a &= (a^*abb^\dagger (a^*abb^\dagger)^\dagger a^*(a^\dagger)^*)^* = a^\dagger aa^*abb^\dagger (a^*abb^\dagger)^\dagger \\ &= a^*abb^\dagger (a^*abb^\dagger)^\dagger \text{ is self}-\text{adjoint}. \end{aligned}$$

Therefore, we show that $(a^\dagger abb^\dagger)^\dagger = (a^*abb^\dagger)^\dagger a^*a$ and then we get, by (f2),

$$(abb^\dagger)^\dagger = (a^*abb^\dagger)^\dagger a^* = (a^*abb^\dagger)^\dagger a^*aa^\dagger = (a^\dagger abb^\dagger)^\dagger a^\dagger.$$

Thus, the condition (f1) holds.

3. Reverse Order Law in C^*-algebras

Now, we consider some additional equivalent conditions for the reverse order law $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ for elements of C^*–algebras. First, we have the following result.

Lemma 3.1. *Let $\mathcal{A}$ be a unital C^*–algebra and let $a, b \in \mathcal{A}^-$. Then the following statements are equivalent:*

(1) $ab \in \mathcal{A}^-$;

(2) $a^\dagger abb^\dagger \in \mathcal{A}^-$;

(3) $(1 - bb^\dagger)(1 - a^\dagger a)\mathcal{A}^-$;

(4) $(a^\dagger)^* b \in \mathcal{A}^-$;

(5) $a(b^\dagger)^* \in \mathcal{A}^-$;

(7) $b^\dagger a^\dagger \in \mathcal{A}^-$;

(8) $(1 - a^\dagger a)(1 - bb^\dagger) \in \mathcal{A}^-$;

(9) $a^\dagger ab \in \mathcal{A}^-$;

(10) $abb^\dagger \in \mathcal{A}^-$.

Proof. Using Theorem 1.1, Theorem 1.2 and Lemma 1.3, we can easy get these equivalences. Notice that the condition $a^\dagger abb^\dagger \in \mathcal{A}^-$ implies $bb^\dagger a^\dagger a = (a^\dagger abb^\dagger)^* \in \mathcal{A}^-$. Since $a^\dagger a, bb^\dagger \in \mathcal{P}(\mathcal{A})$, then, by Lemma 1.3, the condition $a^\dagger abb^\dagger \in \mathcal{A}^\dagger$ is equivalent to $(1 - bb^\dagger)(1 - a^\dagger a) \in \mathcal{A}^\dagger$, that is $a^\dagger abb^\dagger \in \mathcal{A}^- \Leftrightarrow (1 - bb^\dagger)(1 - a^\dagger a) \in \mathcal{A}^-$.

Theorem 3.2. *Let $\mathcal{A}$ be a unital C^*–algebra and let $a, b, ab \in \mathcal{A}^-$. Then the following statements are equivalent:*

(a1) $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$;

(a3) $(ab)^\dagger = b^\dagger a^\dagger - b^\dagger[(1 - bb^\dagger)(1 - a^\dagger a)]^\dagger a^\dagger$;

(b3) $[(a^\dagger)^* b]^\dagger = b^\dagger a^* - b^\dagger[(1 - bb^\dagger)(1 - a^\dagger a)]^\dagger a^*$;

(c3) $[a(b^\dagger)^*]^\dagger = b^* a^\dagger - b^*[(1 - bb^\dagger)(1 - a^\dagger a)]^\dagger a^\dagger$;

(d3) $(b^\dagger a^\dagger)^\dagger = ab - a[(1 - a^\dagger a)(1 - bb^\dagger)]^\dagger b$;

(f3) $(a^\dagger ab)^\dagger = b^\dagger a^\dagger a - b^\dagger[(1 - bb^\dagger)(1 - a^\dagger a)]^\dagger a^\dagger a$ *and* $(abb^\dagger)^\dagger = bb^\dagger a^\dagger - bb^\dagger[(1 - bb^\dagger)(1 - a^\dagger a)]^\dagger a^\dagger$.

Proof. By Lemma 2.1, the hypothesis $ab \in \mathcal{A}^-$ implies regularity of suitable elements. Let (b1), (c1) , (d1), (f1) be conditions from Theorem 2.1. The equivalences (a1) ⇔ (b1) ⇔ (c1) ⇔ (d1) ⇔ (f1) follow from Theorem 2.1.

(a1) ⇔ (a3): Since $a^\dagger a, bb^\dagger \in \mathcal{P}(\mathcal{A})$, then, by Theorem 1.4, we obtain the formula

$$(a^\dagger abb^\dagger)^\dagger = bb^\dagger a^\dagger a - bb^\dagger[(1-bb^\dagger)(1-a^\dagger a)]^\dagger a^\dagger a, \tag{23}$$

which gives the equality

$$\begin{aligned} b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger &= b^\dagger(bb^\dagger a^\dagger a - bb^\dagger[(1-bb^\dagger)(1-a^\dagger a)]^\dagger a^\dagger a)a^\dagger \\ &= b^\dagger a^\dagger - b^\dagger[(1-bb^\dagger)(1-a^\dagger a)]^\dagger a^\dagger \end{aligned} \tag{24}$$

Now, we deduce that $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ if and only if $(ab)^\dagger = b^\dagger a^\dagger - b^\dagger[(1-bb^\dagger)(1-a^\dagger a)]^\dagger a^\dagger$. Therefore, the statement (a1) is equivalent to (a3).

(b1) ⇔ (b3): Multiplying the equality (24) by aa^* from the right side, we get

$$b^\dagger(a^\dagger abb^\dagger)^\dagger a^* = b^\dagger a^* - b^\dagger[(1-bb^\dagger)(1-a^\dagger a)]^\dagger a^*.$$

So, $[(a^\dagger)^* b]^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^*$ and $[(a^\dagger)^* b]^\dagger = b^\dagger a^* - b^\dagger[(1-bb^\dagger)(1-a^\dagger a)]^\dagger a^*$ are equivalent, that is (b1) ⇔ (b3).

(c1) ⇔ (c3): Multiplying the equality (24) by b^*b from the left side, we have

$$b^*(a^\dagger abb^\dagger)^\dagger a^\dagger = b^* a^\dagger - b^*[(1-bb^\dagger)(1-a^\dagger a)]^\dagger a^\dagger$$

which yields this equivalence.

(d1) ⇔ (d3): Using Theorem 1.4, we observe that

$$(bb^\dagger a^\dagger a)^\dagger = a^\dagger abb^\dagger - a^\dagger a[(1-a^\dagger a)(1-bb^\dagger)]^\dagger bb^\dagger.$$

Multiplying this equality by a from the left side and by b from the right side we get

$$a(bb^\dagger a^\dagger a)^\dagger b = ab - a[(1-a^\dagger a)(1-bb^\dagger)]^\dagger b.$$

The equivalence (d1) ⇔ (d3) easy follows.

(f1) ⇔ (f3): Multiplying the equality (23) first by $b^\dagger$ from the left side, we have

$$b^\dagger(a^\dagger abb^\dagger)^\dagger = b^\dagger a^\dagger a - b^\dagger[(1-bb^\dagger)(1-a^\dagger a)]^\dagger a^\dagger a,$$

and then by $a^\dagger$ from the right side, we obtain

$$(a^\dagger abb^\dagger)^\dagger a^\dagger = bb^\dagger a^\dagger - bb^\dagger[(1-bb^\dagger)(1-a^\dagger a)]^\dagger a^\dagger.$$

Now, this part of proof easy follows.

As a consequence of Theorem 1.5 and Theorem 2.1 we get the following result.

Corollary 3.3. *Let $\mathcal{R}$ be a ring with involution and let $a, b \in \mathcal{R}^\dagger$. Then the following statements are equivalent:*

(a1) $ab, a^\dagger abb^\dagger \in \mathcal{R}^\dagger$ *and* $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$;

(e1) $ab, a^\dagger ab, abb^\dagger \in \mathcal{R}^\dagger$ *and* $(ab)^\dagger = (a^\dagger ab)^\dagger a^\dagger = b^\dagger(abb^\dagger)^\dagger$;

(e2) $(a^\dagger)^* b, a^\dagger ab, (a^\dagger)^* bb^\dagger \in \mathcal{R}^\dagger$ *and* $[(a^\dagger)^* b]^\dagger = (a^\dagger ab)^\dagger a^* = b^\dagger[(a^\dagger)^* bb^\dagger]^\dagger$;

(e3) $a(b^\dagger)^*, a^\dagger a(b^\dagger)^*, abb^\dagger \in \mathcal{R}^\dagger$ *and* $[a(b^\dagger)^*]^\dagger = [a^\dagger a(b^\dagger)^*]^\dagger a^\dagger = b^*(abb^\dagger)^\dagger$;

(e4) $b^\dagger a^\dagger, bb^\dagger a^\dagger, b^\dagger a^\dagger a \in \mathcal{R}^\dagger$ *and* $(b^\dagger a^\dagger)^\dagger = (bb^\dagger a^\dagger)^\dagger b = a(b^\dagger a^\dagger a)^\dagger$;

(e5) $ab, a^* ab, abb^* \in \mathcal{R}^\dagger$ *and* $(ab)^\dagger = (a^* ab)^\dagger a^* = b^*(abb^*)^\dagger$;

(e6) $(a^\dagger)^* b, (a^* a)^\dagger b, (a^\dagger)^* bb^* \in \mathcal{R}^\dagger$ *and* $[(a^\dagger)^* b]^\dagger = [(a^* a)^\dagger b]^\dagger a^\dagger = b^*[(a^\dagger)^* bb^*]^\dagger$;

(e7) $a(b^\dagger)^*, a^* a(b^\dagger)^*, a(bb^*)^\dagger \in \mathcal{R}^\dagger$ *and* $[a(b^\dagger)^*]^\dagger = [a^* a(b^\dagger)^*]^\dagger a^* = b^\dagger[a(bb^*)^\dagger]^\dagger$;

(e8) $(b^\dagger a^\dagger)^*, (a^\dagger)^*(bb^*)^\dagger, (a^* a)^\dagger(b^\dagger)^* \in \mathcal{R}^\dagger$ *and* $[(b^\dagger a^\dagger)^\dagger]^* = b^\dagger[(a^\dagger)^*(bb^*)^\dagger]^\dagger = [(a^* a)^\dagger(b^\dagger)^*]^\dagger a^\dagger$.

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